

Analysis of Distributed Randomized Algorithms

Akka Zemmari

LaBRI - Université Bordeaux 1

Séminaire LIF
Apr. 05, 2012

- The Model
- Distributed (Randomized) Algorithms
- The MIS Problem
- The Colouring Problem
 - The Algorithm
 - A Lower Bound

Definitions [Tel]

- A distributed system is an interconnected collection of autonomous computers, processes, or processors. The computers, processes, or processors are referred to as the nodes of the distributed system.
- to be qualified as autonomous, the nodes must at least be equipped with their own private control,
- to be qualified as interconnected, the nodes must be able to exchange information.

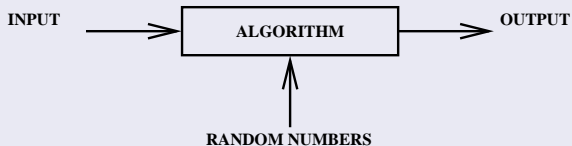
Definitions [Tel]

A distributed algorithm for a collection $P = \{p_1, p_2, \dots, p_n\}$ of processes is a collection of local algorithms, one for each process in P .

Randomized Algorithm [Motwani & Raghavan]

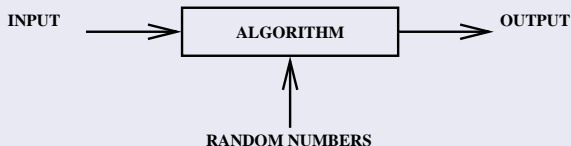


Randomized Algorithm [Motwani & Raghavan]



- In addition to input, algorithm takes a source of random numbers and makes random choices during execution,
- Behavior can vary even on a fixed input.

Algorithme probabiliste [Motwani & Raghavan]



- Design algorithm + analysis to show that this behavior is likely to be good, on every input.
(The likelihood is over the random numbers only.)

Hypothesis

- Anonymous networks
- No global knowledge

(Example of) Impossibility results [Angluin]

- There is no deterministic algorithm that solves the election problem in anonymous networks.

⇒ Randomized distributed algorithms

- [Métivier, Saheb, Z.] Locally guided randomized elections in trees: The totally fair case. *Inf. Comput.* 198(1): 40-55 (2005)
- [Marckert, Saheb, Z.] Election Algorithms with Random Delays in Trees. In 21st International Conference on Formal Power Series and Algebraic Combinatoric. (2009)
- [El Hibaoui, Robson, Saheb, Z.] Uniform election in trees and polyominoids. *Discrete Applied Mathematics* 158(9): 981-987 (2010)

Definitions

- *Time Complexity.* [1] A round (cycle) of each processor is composed of the following three steps: 1. Send messages to (some of) the neighbours, 2. Receive messages from (some of) the neighbours, 3. Perform some local computation. As usual, the time complexity is the maximum possible number of rounds needed until every node has completed its computation.
- *Bit Complexity.* [2] A bit round is a round such that each processor can send/receive at most 1 bit to/from each neighbour. The bit complexity of an algorithm \mathcal{A} is the number of bit rounds to complete algorithm \mathcal{A} .

- [1] D. Peleg: *Distributed computing - A Locality-sensitive approach.* (2000).
- [2] K. Kothapalli and M. Onus and C. Scheideler and C. Schindelhauer: *Distributed coloring in $O(\sqrt{\log n})$ bit rounds.* (IPDPS) (2006).

Definitions

- *Time Complexity.* [1] A round (cycle) of each processor is composed of the following three steps: 1. Send messages to (some of) the neighbours, 2. Receive messages from (some of) the neighbours, 3. Perform some local computation. As usual, the time complexity is the maximum possible number of rounds needed until every node has completed its computation.
- *Bit Complexity.* [2] A bit round is a round such that each processor can send/receive at most 1 bit to/from each neighbour. The bit complexity of an algorithm \mathcal{A} is the number of bit rounds to complete algorithm \mathcal{A} .

- [1] D. Peleg: *Distributed computing - A Locality-sensitive approach.* (2000).
- [2] K. Kothapalli and M. Onus and C. Scheideler and C. Schindelhauer: *Distributed coloring in $O(\sqrt{\log n})$ bit rounds.* (IPDPS) (2006).

The MIS Problem

Definition

Let $G = (V, E)$ be a graph. An independent set is a subset U of V s.t.,

$$\forall u, v \in U, \{u, v\} \notin E,$$

U is maximal if $\forall v \in V \setminus U, U \cup \{v\}$ is not an independent set.



The MIS Problem: Related Works

	Knowledge	Time (on average)	Message size (number of bits)	Bit complexity
Luby (Lynch)	Size of the graph	$O(\log n)$	$\log n$	$O(\log^2 n)$
Luby (Peleg)	Maximum degree in 2-Neighbourhood	$O(\log^2 n)$	$\log n$	$O(\log^3 n)$
Luby (Wattenhofer)	Maximum of neighbours degrees	$O(\log n)$	$\log n$	$O(\log^2 n)$
Alon, Babai and Itai	Maximum of neighbours degrees	$O(\log n)$	$\log n$	$O(\log^2 n)$
Métivier et al. [1]	no knowledge	$O(\log n)$	1	$O(\log n)$
Métivier et al. [2]	no knowledge		1	$\Omega(\sqrt{\log n})$
Métivier et al. [3]			1	$O(\sqrt{\log n})$ in ring graphs.

- [1] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari. *An optimal bit complexity randomized distributed MIS algorithm*. Distributed Computing (2011).
- [2] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari. *About Lower Bounds on the Bit Complexity and the Execution Time of some Probabilistic Distributed Graph Algorithms*. Submitted.
- [3] A. Fontaine, Y. Métivier, J.M. Robson, A. Zemmari *Optimal Bit Complexity Randomised Distributed MIS and Maximal Matching Algorithms for Anonymous Rings*. Submitted.

The Colouring Problem

Definition.

Let $G = (V, E)$ be a simple connected undirected graph. A proper vertex colouring for G is an assignment of a colour $c(v)$ to each vertex v , such that any two adjacent vertices have a different colour, i.e., $c(v) \neq c(u)$ for every $\{u, v\} \in E$.



	Time	Bit Complexity	Observations
Johansson [1]	$O(\log n)$	$O(\log^2 n)$	$\Delta + 1$ colours
Kothapalli et al. [2]		$\Omega(\log n)$	no restriction on the number of colours
Métivier et al. [3]	$O(\log n)$	$O(\log n)$	$\Delta + 1$ colours
Métivier et al. [4]		$\leq 2 \log n + o(\log n)$ $> 2 \log n$	3-colouring of ring graphs

- [1] Ö. Johansson: *Simple distributed $(\Delta + 1)$ -coloring of graphs*. IPL. (1999).
- [2] K. Kothapalli and M. Onus and C. Scheideler and C. Schindelhauer: *Distributed coloring in $O(\sqrt{\log n})$ bit rounds*. IPDPS. (2006).
- [3] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *About randomised distributed graph colouring and graph partition algorithms*. Inf. Comput. (2010).
- [4] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *On the Time and the Bit Complexity of Distributed Randomised Anonymous Ring Colouring*. TCS (2012).

	Time	Bit Complexity	Observations
Johansson [1]	$O(\log n)$	$O(\log^2 n)$	$\Delta + 1$ colours
Kothapalli et al. [2]		$\Omega(\log n)$	no restriction on the number of colours
Métivier et al. [3]	$O(\log n)$	$O(\log n)$	$\Delta + 1$ colours
Métivier et al. [4]		$\leq 2 \log n + o(\log n)$ $> 2 \log n$	3-colouring of ring graphs

- [1] Ö. Johansson: *Simple distributed $(\Delta + 1)$ -coloring of graphs*. IPL. (1999).
- [2] K. Kothapalli and M. Onus and C. Scheideler and C. Schindelhauer: *Distributed coloring in $O(\sqrt{\log n})$ bit rounds*. IPDPS. (2006).
- [3] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *About randomised distributed graph colouring and graph partition algorithms*. Inf. Comput. (2010).
- [4] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *On the Time and the Bit Complexity of Distributed Randomised Anonymous Ring Colouring*. TCS (2012).

	Time	Bit Complexity	Observations
Johansson [1]	$O(\log n)$	$O(\log^2 n)$	$\Delta + 1$ colours
Kothapalli et al. [2]		$\Omega(\log n)$	no restriction on the number of colours
Métivier et al. [3]	$O(\log n)$	$O(\log n)$	$\Delta + 1$ colours
Métivier et al. [4]		$\leq 2 \log n + o(\log n)$ $> 2 \log n$	3-colouring of ring graphs

- [1] Ö. Johansson: *Simple distributed $(\Delta + 1)$ -coloring of graphs*. IPL. (1999).
- [2] K. Kothapalli and M. Onus and C. Scheideler and C. Schindelhauer: *Distributed coloring in $O(\sqrt{\log n})$ bit rounds*. IPDPS. (2006).
- [3] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *About randomised distributed graph colouring and graph partition algorithms*. Inf. Comput. (2010).
- [4] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *On the Time and the Bit Complexity of Distributed Randomised Anonymous Ring Colouring*. TCS (2012).

	Time	Bit Complexity	Observations
Johansson [1]	$O(\log n)$	$O(\log^2 n)$	$\Delta + 1$ colours
Kothapalli et al. [2]		$\Omega(\log n)$	no restriction on the number of colours
Métivier et al. [3]	$O(\log n)$	$O(\log n)$	$\Delta + 1$ colours
Métivier et al. [4]		$\leq 2 \log n + o(\log n)$ $> 2 \log n$	3-colouring of ring graphs

- [1] Ö. Johansson: *Simple distributed $(\Delta + 1)$ -coloring of graphs*. IPL. (1999).
- [2] K. Kothapalli and M. Onus and C. Scheideler and C. Schindelhauer: *Distributed coloring in $O(\sqrt{\log n})$ bit rounds*. IPDPS. (2006).
- [3] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *About randomised distributed graph colouring and graph partition algorithms*. Inf. Comput. (2010).
- [4] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *On the Time and the Bit Complexity of Distributed Randomised Anonymous Ring Colouring*. TCS (2012).

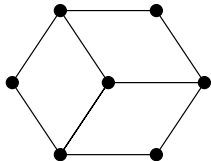
	Time	Bit Complexity	Observations
Johansson [1]	$O(\log n)$	$O(\log^2 n)$	$\Delta + 1$ colours
Kothapalli et al. [2]		$\Omega(\log n)$	no restriction on the number of colours
Métivier et al. [3]	$O(\log n)$	$O(\log n)$	$\Delta + 1$ colours
Métivier et al. [4]		$\leq 2 \log n + o(\log n)$ $> 2 \log n$	3-colouring of ring graphs

- [1] Ö. Johansson: *Simple distributed $(\Delta + 1)$ -coloring of graphs*. IPL. (1999).
- [2] K. Kothapalli and M. Onus and C. Scheideler and C. Schindelhauer: *Distributed coloring in $O(\sqrt{\log n})$ bit rounds*. IPDPS. (2006).
- [3] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *About randomised distributed graph colouring and graph partition algorithms*. Inf. Comput. (2010).
- [4] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *On the Time and the Bit Complexity of Distributed Randomised Anonymous Ring Colouring*. TCS (2012).

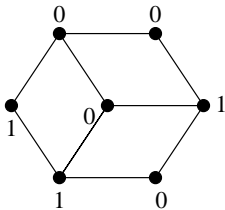
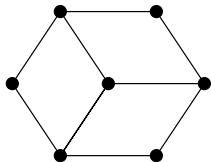
	Time	Bit Complexity	Observations
Johansson [1]	$O(\log n)$	$O(\log^2 n)$	$\Delta + 1$ colours
Kothapalli et al. [2]		$\Omega(\log n)$	no restriction on the number of colours
Métivier et al. [3]	$O(\log n)$	$O(\log n)$	$\Delta + 1$ colours
Métivier et al. [4]		$\leq 2 \log n + o(\log n)$ $> 2 \log n$	3-colouring of ring graphs

- [1] Ö. Johansson: *Simple distributed $(\Delta + 1)$ -coloring of graphs*. IPL. (1999).
- [2] K. Kothapalli and M. Onus and C. Scheideler and C. Schindelhauer: *Distributed coloring in $O(\sqrt{\log n})$ bit rounds*. IPDPS. (2006).
- [3] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *About randomised distributed graph colouring and graph partition algorithms*. Inf. Comput. (2010).
- [4] Y. Métivier, J. M. Robson, N. Saheb-Djahromi, A. Zemmari: *On the Time and the Bit Complexity of Distributed Randomised Anonymous Ring Colouring*. TCS (2012).

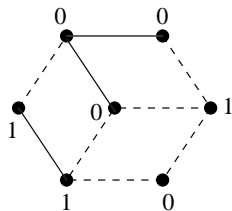
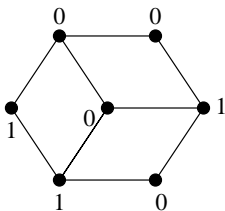
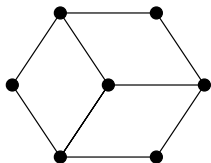
Algorithm *Fast_Colour*



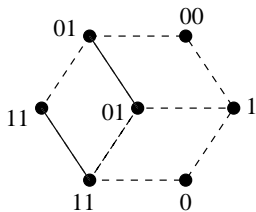
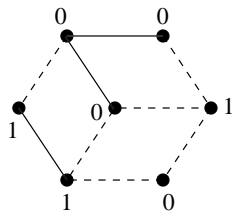
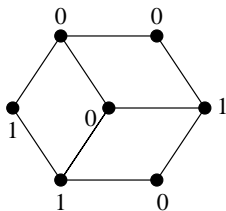
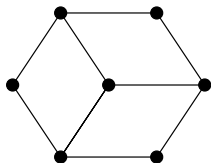
Algorithm *Fast_Colour*



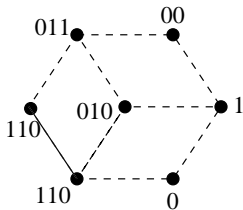
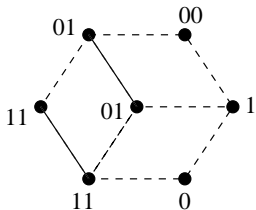
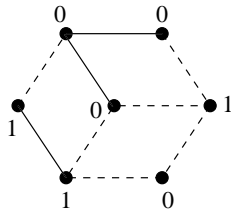
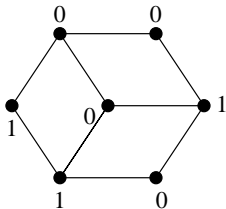
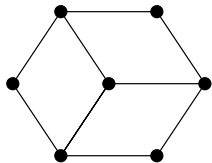
Algorithm *Fast_Colour*



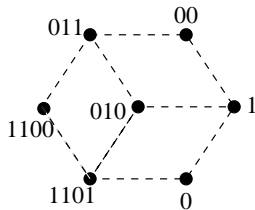
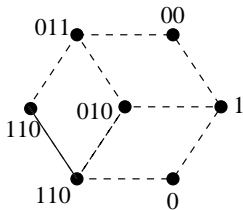
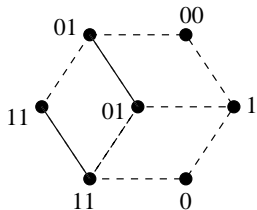
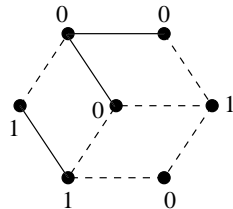
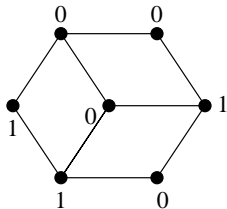
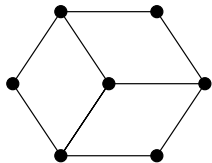
Algorithm *Fast_Colour*



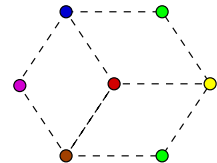
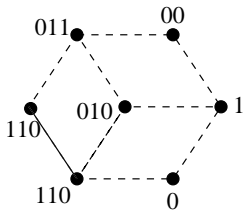
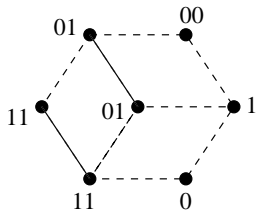
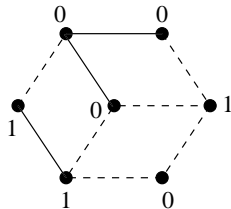
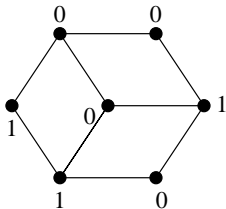
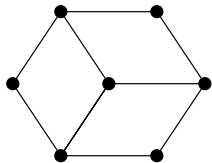
Algorithm *Fast_Colour*



Algorithm *Fast_Colour*



Algorithm *Fast_Colour*



Lemma.

In any phase of the algorithm, the expected number of edges removed from the residual graph G is half the number of its edges.

Corollary.

There are constants k_1 and K_1 such that for any graph G of n vertices, the number of phases to remove all edges from G is:

- less than $k_1 \log n$ on average,
- less than $K_1 \log n$ w.h.p.

Lemma.

In any phase of the algorithm, the expected number of edges removed from the residual graph G is half the number of its edges.

Corollary.

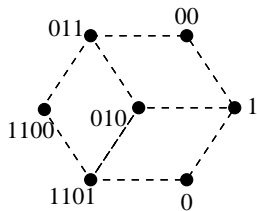
There are constants k_1 and K_1 such that for any graph G of n vertices, the number of phases to remove all edges from G is:

- less than $k_1 \log n$ on average,
- less than $K_1 \log n$ w.h.p.

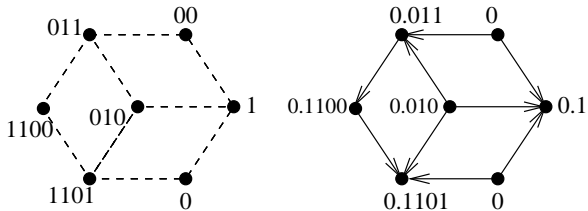
Theorem.

Algorithm *Fast_Colour* computes a colouring for any arbitrary graph of size n in time $O(\log n)$ w.h.p., each message containing 1 bit.

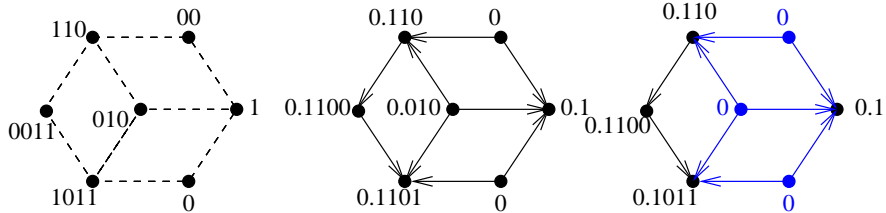
Reducing the Number of Colors - Algorithm \mathcal{R}



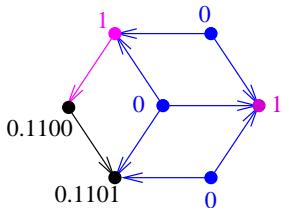
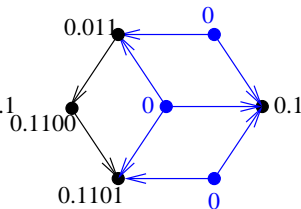
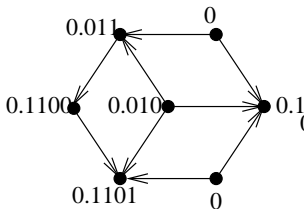
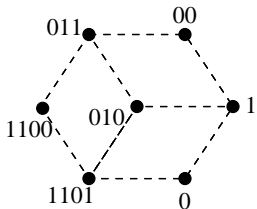
Reducing the Number of Colors - Algorithm \mathcal{R}



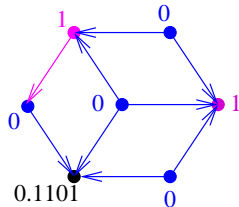
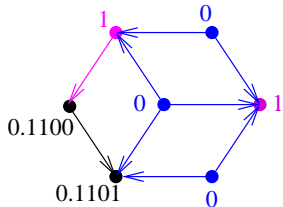
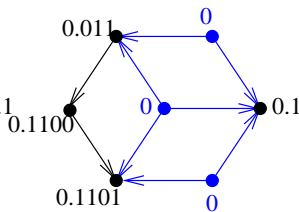
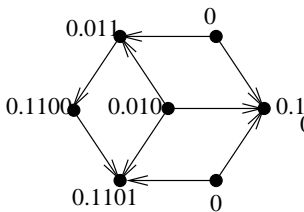
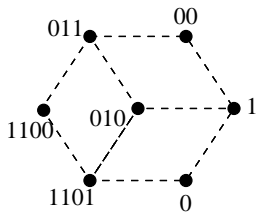
Reducing the Number of Colors - Algorithm \mathcal{R}



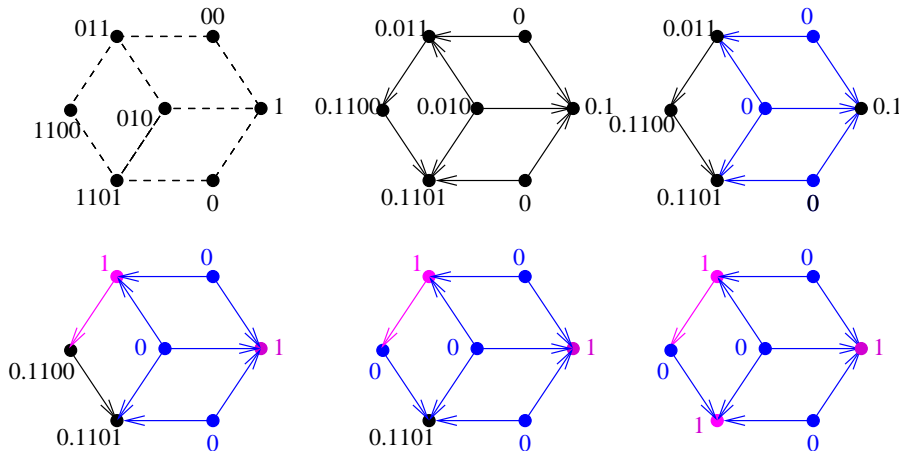
Reducing the Number of Colors - Algorithm \mathcal{R}



Reducing the Number of Colors - Algorithm \mathcal{R}



Reducing the Number of Colors - Algorithm \mathcal{R}



Theorem

For any graph $G = (V, E)$ having a maximum degree Δ , Algorithm \mathcal{R} achieves a $(\Delta + 1)$ -colouring of G in at most $e\Delta + 2 \log n + \log^* n$ rounds w.h.p.

A Particular Case and a Lower Bound

Theorem

Let $G = (V, E)$ be a ring of size $n \geq 3$. Let T_1 denote the number of rounds necessary to colour all vertices by the colouring algorithm.

- The expected value of T_1 is asymptotically equal to $\log_2 n - \frac{1}{2} + \frac{\gamma}{\log 2} + Q(\log_2 n) + O(n^{-2})$, where $Q(u) = -\frac{1}{\log 2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \Gamma\left(\frac{2ik\pi}{\log 2}\right) e^{-2ik\pi u}$ is a Fourier series with period 1 and with an amplitude which does not exceed 10^{-6} .
- it is less than $2 \log_2 n$ w.h.p.

Lemma

The time complexity of the decreasing colours algorithm is less than $e + 2 \log_2 n$ w.h.p.

Time Bounds for the 3–Colouring Algorithm

Theorem

The total time for the 3–colouring algorithm is $\log_2 n + o(\log n)$ on average and $2 \log_2 n + o(\log n)$ w.h.p.

Theorem

- *A Las Vegas algorithm A which computes a proper colouring on rings takes time at least $\log_2 n - o(\log n)$ on average.*
- *No Las Vegas algorithm A which computes a proper colouring on rings runs in time $k \log_2 n$ with high probability for any $k < 2$.*

Formal proof of distributed randomised algorithms (joint work with A. Fontaine and P. Castéran)

- Formal modelisation in Coq.
- Proving correctness of such algorithms..

Thank You