Probabilistic analysis of a reinforcement-learning algorithm for finding shortest paths

Zoé Varin
ENS de Lyon

Under the supervision of **Daniel Kious** and **Cécile Mailler**
University of Bath

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Abstract

We study a slight variation of the model defined in [KMS20], in which ants look for shortest paths between their nest and a source of food. On a graph G with three marked nodes N_1, N_2 and F (two nests and the food), we study the following model : at step n, the n-th ant does a random walk which starts at a random nest N_i, and is stopped when first hitting F. The random walk at step n has transition probabilities that depend on the previous realisations before step n: for each n, the n-th ant, on its way back, reinforces a subset of the edges it has crossed on its way forward to the food; then at each step of the n-th random walk, the probability to choose an edge is proportional to its weight at the end of step $n-1$.

We focus on the *loop-erased* reinforcement model, where ants reinforce edges they cross while going back from the food to their nest, avoiding useless loops. We prove that on the generalised triangle (where each edge of a triangle has been replaced by a path) the normalised edge-weights converge almost surely, and we give the limiting values according to the lengths of each path.

1 Introduction

In biology, it has long been known that ants use pheromones to communicate, laying *trail pheromones* on the ground, that the next ants can read, and it helps them find food efficiently. They are known to be able to find shortest paths from their nest to a source of food. It is possible to model this as a set of ants doing simultaneously a series of random walks on a graph with two marked nodes, a nest and a source of food, each random walk starting at the nest and being stopped when first reaching the food; then the ants go back, and do it again. Then the deposition of pheromones by ants can be seen as the real equivalent of the reinforcement of some weights of some useful edges after each random walk. One can for example read [DS04, Chapter 1] to have a more detailed overview of the path that started by the observation of different species of ants and led to the creation of *ant colony optimization* (ACO) algorithms.

In their papers, Kious, Mailler and Schapira ([KMS20], [KMS21]) introduced a probabilistic reinforcement-learning model inspired by the real-life ants, but slightly different from ACO. For a graph G with two marked nodes, a nest N, a source of food F, and edges having weight 1 at the beginning, they consider the following process. At each time-step $n \geq 1$, an ant starts a random walk from N, and stops when it first reaches F. The n-th ant's random walk depends on the reinforcement of the edge-weights after the previous excursions: at each time step, after their excursion, the ant reinforces a subset of the edges it has crossed (i.e. deposits pheromones on it), increasing by 1 the weights of the corresponding edges. Then, at each step of the n-th ant's random walk, the probability to cross an edge is proportional to its weight at time $n-1$.

Kious, Mailler and Schapira ([KMS20], [KMS21]) showed that the behaviour of the process deeply depends on the choice of the subset to reinforce at each step . In the *trace-reinforced model* (T), they reinforce all the edges that have been crossed at least once. For the two other models

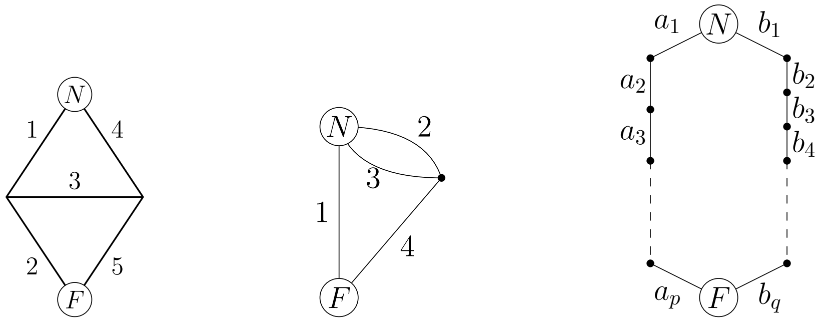


Figure 1: From left to right, the lozenge, the cone, and the (p, q)-path, studied in [KMS21]. In the model (T), all the edges except edge 4 on the cone have a normalised weight that do not converge to 0.

they propose, following the ants analogy, we reinforce edges corresponding to the path taken by the ant on its way back to the nest. In the *loop-erased model* (LE), the ants walk backwards, but avoiding useless loops, whereas in the *geodesic model* (G), they take one of the shortest paths they know, i.e. one of the shortest paths within the trace of their forward excursion from N to F.

A natural question is now: do the ants, following one of these models, asymptotically find a shortest path from N to F? More precisely, does the proportion of ants that go from N to F using a shortest path almost surely tend to 1 when time goes to infinity?

The (LE) and (G) models are promising, and it has been conjectured in [KMS20] that in those two models the ants find a shortest path. They proved that as soon as the graph G is a series-parallel graph, the ants indeed find a shortest path in the (LE) model [KMS20], whereas it is also true in the (G) model for a special lozenge graph. The (T) model is a bit less efficient (intuitively, the ants learn less, and in fact this is what happend), and even if the ants find shortest path if G belongs to a restricted class of graphs (if G is such that N and F are at distance 1 and is a *tree-like graphs*, i.e. $G \backslash F$ is a tree), there are several simple graphs on which the ants keep reinforcing edges that do not belong to a geodesic (see the examples of the cone, the (p, q)-path and the lozenge in [KMS21], illustrated on Figure 1).

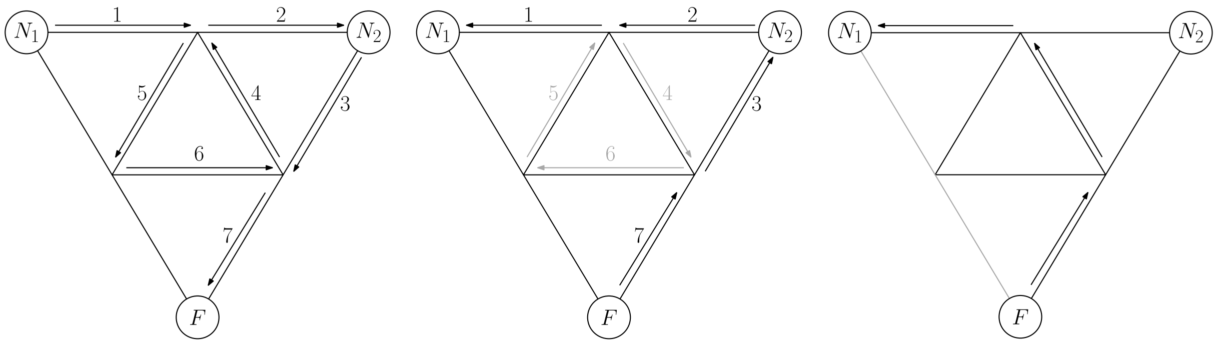
In this paper, we consider this model, but instead of having one marked node N, the nest at each step $N(n)$ is chosen randomly between two nests. We focus on the more promising models, the (LE) and (G) ones.

Mathematical description of the 2-nest model. Our model is almost the same as the model defined in [KMS20], with only a slight difference (which however changes considerably the behaviour of the model): the starting nest is random.

We consider a graph $G = (V, E)$ with three marked nodes, two nests N_1 and N_2, and a source of food, F. We also let α_1 and α_2 be two real numbers such that $\alpha_1 > 0$, $\alpha_2 > 0$, and $\alpha_1 + \alpha_2 = 1$. Each α_i corresponds to the size of the nest N_i. We then consider the following process. At time 0, we have weights $W_e(0) = 1$ on every edge $e \in E$, and $W_e(0) = 0, \forall e \notin E$. Then at each step n, an ant does a random walk from a random nest $N(n)$ to the source of food F. More precisely, $N(n) = N_1$ with probability α_1, and $N(n) = N_2$ with probability α_2Moreover, the random walk is weighted by $W(n-1)$ (where we use the notation $W(n) = (W_e(n))_{e \in E}$), and is stopped when first hitting F. We thus define the random walk $X^{(n)}$ in the following way,

- $X_0^{(n)} = N(n)$,
- $\forall i \geq 0, \mathbb{P}\left(X_{i+1}^{(n)} = F \middle| X_i^{(n)} = F, W(n-1)\right) = 1$,
- $\forall i \geq 0, \forall u \neq F, \forall v,\ \mathbb{P}\left(X_{i+1}^{(n)} = u \middle| X_i^{(n)} = v, W(n-1)\right) = \frac{W_{uv}(n-1)}{\sum_{w \sim v} W_{vw}(n-1)}$.

We then reinforce a subset of the edges which have been used by the nth ant. Here we have several ways to do that (which are illustrated on Figure 2):



(a) **(b)** **(c)**

Figure 2: Illustration of the reinforcement process. Subfigure 2a shows the trajectory of an ant starting at N_1. Subfigure 2b illustrates the edges taken by this ant avoiding loops on its way back to N_1, as in the loop-erased(LE) model. Finally, Subfigure 2c illustrates the path taken by the ant in the geodesic (G) model, which corresponds to the shortest path on the subgraph explored by the ant on its way forward. One can notice that the geodesic reinforcement can give, as in this example, a backward path that is strictly shorter.

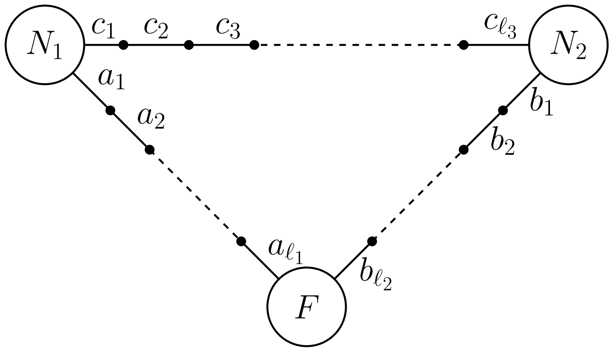


Figure 3: The (ℓ_1, ℓ_2, ℓ_3)-triangle.

- In the *loop-erased model* (LE), we reinforce edges that would be taken by the nth ant going back to the nest and avoiding useless loops it had done on its way forward. Formally, we define $i_0 := \min\{k : X_k^{(n)} = F\}$, and while j is such that $X_{i_j}^{(n)} \neq N(n)$, we define $i_{j+1} := \min\{k-1 : X_k^{(n)} = X_{i_j}^{(n)}\}$. We then define the subset of edges to be reinforced: $\gamma^{(n)} := \{X_{i_{j+1}}^{(n)} X_{i_j}^{(n)}, 0 \leq j < J\}$, where J is the largest j for which i_j has been defined.

- In the *geodesic model* (G), we let $\gamma^{(n)}$ denote the edges of one of the shortest paths from $N(n)$ to F in the subgraph explored by the nth ant, $G^{(n)} = (V, \cup_{0 \leq i < i_0} \{X_i^{(n)} X_{i+1}^{(n)}\})$, chosen uniformly at random (where i is again defined as $i_0 := \min\{k : X_k^{(n)} = F\}$).

In both cases, we reinforce the edges that belong to $\gamma^{(n)}$:

$$\forall e \in E, W_e(n+1) = W_e(n) + \mathbb{1}_{e \in \gamma^{(n)}}.$$

Main result. We now state our main results, which gives the convergence of the normalised weights in the case when the graph G is a subdivided triangle as on Figure 3.

For all three integers ℓ_1, ℓ_2 and ℓ_3, we define the (ℓ_1, ℓ_2, ℓ_3)*-triangle* as in Figure 3: it is a graph composed of only one cycle, with ℓ_1 (respectively ℓ_2) edges between N_1 (respectively N_2) and F, and ℓ_3 edges between N_1 and N_2, named as on Figure 3. We define the following weights: $W_1(n) := W_{a_1}(n)$, $W_2(n) := W_{b_1}(n)$ and $W_3(n) := W_{c_1}(n)$. Our first result states that all the weights between two marked nodes (N_1, N_2 and F) are almost surely equal, and that for every edge, the sequence of its normalised weights has a limit, which can be computed:

Theorem 1. *If G is a (ℓ_1,ℓ_2,ℓ_3)-triangle then, both in the (LE) reinforcement model and in the (G) one, the sequence of weights is such that, almost surely, for all n: $\forall i \leq \ell_1, W_{a_i}(n) = W_{a_1}(n)$, $\forall i \leq \ell_2, W_{b_i}(n) = W_{b_1}(n)$, and $\forall i \leq \ell_3, W_{c_i}(n) = W_{c_1}(n)$.*

Moreover, the sequence of the normalised weights converges almost surely. The limiting value depends on (ℓ_1,ℓ_2,ℓ_3); without loss of generality, we assume that $\ell_1 \leq \ell_2$ (the other cases can then be solved by symmetry).

- *if $\ell_2 \geq \ell_1 + \ell_3$, then almost surely: $\frac{W_1(n)}{n} \underset{n\to\infty}{\to} 1$, $\frac{W_2(n)}{n} \underset{n\to\infty}{\to} 0$ and $\frac{W_3(n)}{n} \underset{n\to\infty}{\to} \alpha_2$*
- *if $\ell_2 < \ell_1 + \ell_3$ and $\ell_3 < \ell_1 + \ell_2$, then almost surely: $\frac{W_1(n)}{n} \underset{n\to\infty}{\to} \beta_1$, $\frac{W_2(n)}{n} \underset{n\to\infty}{\to} 1-\beta_1$ and $\frac{W_3(n)}{n} \underset{n\to\infty}{\to} \beta_3$, where*

$$\beta_1 = \frac{\alpha_1\ell_1\left(\ell_3+\ell_2-\ell_1\right)}{\ell_1\ell_3 + (\ell_2-\ell_1)\left((1-\alpha_1)(\ell_3-\ell_2)+\alpha_1\ell_1\right)}$$

  and

$$\beta_3 = \frac{\alpha_1\ell_3(1-\alpha_1)(\ell_1+\ell_2-\ell_3)}{(\ell_2-\ell_1)(\ell_1+\ell_2-\ell_3)\alpha_1 + \ell_2(\ell_1-\ell_2+\ell_3)}.$$

- *otherwise, $\ell_3 \geq \ell_1 + \ell_2$, and then almost surely: $\frac{W_1(n)}{n} \underset{n\to\infty}{\to} \alpha_1$, $\frac{W_2(n)}{n} \underset{n\to\infty}{\to} \alpha_2$ and $\frac{W_3(n)}{n} \underset{n\to\infty}{\to} 0$*

Remark 1. *The first part of the theorem, stating that all the weights between N_1 and F (respectively between N_2 and F, and between N_1 and N_2) are equal, is quite obvious since in the (LE) and (G) models, we reinforce only a path from one of the sources to F, and there are only two paths from N_1 (or N_2) to F, and a_i (resp. b_i, c_i) belongs to it if and only if a_1 (resp. b_1, c_1) does.*

Discussion. On the (ℓ_1,ℓ_2,ℓ_3)-triangle, the 2-nest model is already enlightening. When $\ell_3 \geq \ell_1 + \ell_2$ or $\ell_2 > \ell_1 + \ell_3$ the result is simple: the normalised weight of an edge tends to the proportion of ants that have a shortest path from their nest to the food, containing this edge.

When $\ell_2 = \ell_3 + \ell_1$, the theorem highlights an interesting behavior of the ants: the ants starting from N_2 have two shortest paths to F, only one of them sharing edges with the shortest path from N_1 to F (those paths are $ca \coloneqq c_{\ell_3}\dots c_1a_1\dots a_{\ell_1}$ and $b \coloneqq b_1\dots b_{\ell_2}$). If no ant were starting from N_1, then all the normalised weights would converge, but the limit would not be deterministic (and every edge would almost surely have a non-zero limit). Here the ants starting from N_1 influence those starting from N_2, and thus only the normalised weights of the edges belonging to ca have a non-zero limit.

When $\ell_2 < \ell_1 + \ell_3$ and $\ell_3 < \ell_1 + \ell_2$, the result is more intricate. The nests are relatively close to each other, so the ants influence each other, even though it implies a seeming loss of efficiency: for example, on the simple example when $\ell_1 = \ell_2 = \ell_3$, then $\beta_1 = \alpha_1$ and $\beta_3 = \alpha_1\alpha_2$. It means that the edges between N_1 and N_2 keep being reinforced a proportion $\alpha_1\alpha_2$ of the time, even if those edges do not belong to any shortest path from a nest to F. It is not that surprising though. As we said the ants influence each other, and thus tend to create a kind of transport network.

The convergence of this process raises a similar question to the one in the model with one nest: does it converge regardless of the underlying graph ?

In [KMS20], the authors conjectured that, for the (LE) model with only one nest, the normalised edge-weights converge, and the limiting value corresponding to an edge is non-zero if and only if it belongs to at least one shortest path from the nest to the source of food. They proved this result for the class of series-parallel graphs, and conjectured that it is true for every graph.

Here in the 2-nest model, there can be an edge that do not belong to a geodesic from some N_i to F, but have a non-zero limit (recall the case of edges c_is on the (ℓ_1,ℓ_2,ℓ_3)-triangle, when $\ell_2 < \ell_1 + \ell_3$ and $\ell_3 < \ell_1 + \ell_2$). But it seems that such an edge belongs to a geodesic between the

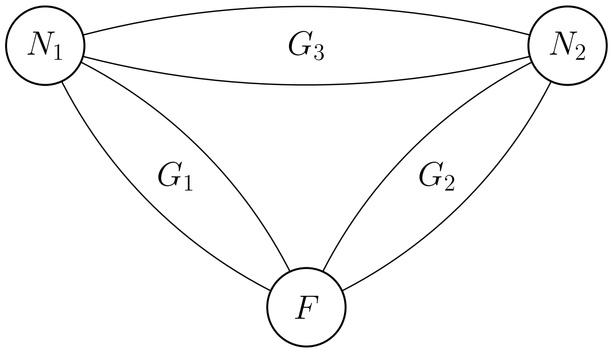


Figure 4: The gluing of three series-parallel graphs G_1, G_2 and G_3 in a triangle fashion.

two nests. It would be interesting to know whether in the 2-nest model, the weight of an edge converge if and only if it belongs to a geodesic between two of the three vertices N_1, N_2 and F.

A first step in this direction, would be to prove it for the family of graphs illustrated on Figure 4 (gluing of three series-parallel graphs in a triangle fashion). Preliminary work suggests that it indeed holds.

2 Strategy and preliminary results

Before diving into the technical details, we give an idea of the general strategy to prove Theorem 1.

Our proof relies on the fact that the renormalised edge-weight process is a stochastic approximation. We give the definition later in this section, we only focus on the main idea for the moment: there is a function F and some "nice" random vector $\xi(n)$ such that

$$\frac{W(n+1)}{n+3} = \frac{W(n)}{n+2} + \frac{1}{n+2}\left(F\left(\frac{W(n)}{n+2}\right) + \xi(n+1)\right).$$

We will see that in fact here the normalised process $\left(\frac{W(n)}{n}\right)_n$ follows the flow of the ODE $\dot{y} = F(y)$. We will prove that any solution to this ODE, regardless of its starting point, converges to some stable zero of F. To get rid of unstable zeros, we will do some couplings with one-dimensional processes called G-urns, which are a generalisation of Pólya's urn processes.

We now define some useful notation, and give an overview of general results that are useful to prove Theorem 1. Those are adapted from [KMS21], in which they were introduced to deal with the case when there is only one nest.

First of all, we define the normalised weights: $\forall e \in E$, $\hat{W}_e(n) \coloneqq \frac{W_e(n)}{n+2}$ and $\hat{W}(n) \coloneqq (\hat{W}_e(n))_{e\in E}$.

In general, we could take any constant c and normalise $\hat{W}_e(n) \coloneqq \frac{W_e(n)}{n+c}$, it would not change the asymptotic behaviour of $\hat{W}$. We choose c to be the number of edges that have F as an endpoint (which is always 2 in what follows), because in this way the weights $(\hat{W}_e(n))_{e\in E}$ lie on a simplex: $\forall n, \sum_{e:F\in e} W_e(n) = n + |\{e : F \in e\}|$, so $c \coloneqq |\{e : F \in e\}|$ implies that $\forall n, \sum_{e:F\in e} \hat{W}_e(n) = 1$.

For the (ℓ_1, ℓ_2, ℓ_3)-triangle, we always have $\hat{W}_2(n) = 1 - \hat{W}_1(n)$, and thus we often work only on $(\hat{W}_1(n), \hat{W}_3(n))$.

We also define, for every edge $e \in E$, and for every $w \in [0,1]^E$:

$$p_e(w) \coloneqq \mathbb{P}\left(W_e(n+1) = W_e(n) + 1 \,\middle|\, \hat{W}(n) = w\right),$$

and

$$F_e(w) \coloneqq p_e(w) - w_e.$$

Finally, we let

$$\begin{array}{rccl} F: & [0,1]^2 & \to & [0,1]^2 \\ & (w_1,w_3) & \longmapsto & (F_1(w_1,w_3), F_3(w_1,w_3))\,. \end{array}$$

To prove convergence of our process $(\hat{W}(n))_{n\geq 0}$, we will use two main tools : G-urn processes, and the ODE method on stochastic approximations. We give definitions and useful results about those tools in the following two subsections.

2.1 Generalised Pólya urn process

Our goal is to show that our process dominates (or is dominated by) some generalised Pólya urn process, and then to use well-known results on these processes. More precisely, we consider processes called G-urn processes: or some function $G:[0,1]\to[0,1]$, $(X_n)_n$ is a *G-urn process* if almost surely, $X_0=1$ and for every n, $X_{n+1}\in\{X_n, X_n+1\}$ and $\mathbb{P}(X_{n+1}=X_n+1|X_0,\ldots,X_n)=G(\hat{X}_n)$.

The following results are stated and proven in this form in [KMS21], and were originally from [Pem07].

Proposition 2 ([Pem07] and Proposition 2.1 in [KMS21]). *Let $(X_n)_{n\geq 0}$ be a G-urn process, with G a $\mathcal{C}^1$-function. Then almost surely $(X_n)_{n\geq 0}$ converges towards a stable fixed point of G, that is a (possibly random) point $p\in[0,1]$, such that $G(p)=p$ and $G'(p)\leq 1$.*

In particular if there exists $c>0$, such that $G(x)>x$, for all $x\in(0,c)$ (resp. $G(x)<x$ for all $x\in(1-c,1)$), then almost surely $\liminf_{n\to\infty} X_n\geq c$ (resp. $\limsup_{n\to\infty} X_n\geq 1-c$).

Corollary 3 (Corollary 2.2 in [KMS21]). *Let $(X_n)_{n\geq 0}$ be an integer valued process adapted to some filtration $(\mathcal{F}_n)_{n\geq 0}$, such that almost surely for all $n\geq 0$, $X_{n+1}\in\{X_n,X_n+1\}$, $X_0=1$, and for some function $G:[0,1]\to[0,1]$,*

$$\mathbb{P}(X_{n+1}=X_n+1|\mathcal{F}_n)\geq G(\hat{X}_n)$$

(where $\forall n, \hat{X}_n=\frac{X_n}{n+2}$). If there exist $c>0,\varepsilon>0$, such that $G(x)>(1+\varepsilon)x$, for all $x\in(0,c)$, then almost surely $\liminf_{n\to\infty}\hat{X}_n\geq c$.

Similarly, if G is such that $\mathbb{P}(X_{n+1}=X_n+1|\mathcal{F}_n)\leq G(\hat{X}_n)$ and there exist $c>0,\varepsilon>0$ such that $G(x)<(1-\varepsilon)x$, for all $x\in(1-c,1)$, then almost surely $\limsup_{n\to\infty}\hat{X}_n\leq 1-c$.

2.2 The ODE method on stochastic approximations

The second major tool we are going to use to prove convergence results is the comparison of our processes to deterministic processes ruled by a "nice" ODE. One can define the notion of stochastic approximation, as it was done in [KMS21]:

A *stochastic approximation* is a process $(X_n)_{n\geq 0}$, adapted to some filtration $(\mathcal{F}_n)_{n\geq 0}$, with values in a convex compact subset $\mathcal{E}\subseteq\mathbb{R}^d$, for some $d\geq 1$, that satisfies an equation of the type

$$X_{n+1}=X_n+\frac{F(X_n)+\xi_{n+1}+r_n}{n+1},\ \text{for all } n\geq 0,$$

where the vector field $F:E\mapsto\mathbb{R}$ is some Lipschitz function, the noise ξ_{n+1} is $\mathcal{F}_{n+1}$-measurable and satisfies $\mathbb{E}_n[\xi_{n+1}]=0$, for all $n\geq 0$, and the remainder term r_n is $\mathcal{F}_n$-measurable and satisfies almost surely $\|r_n\|\leq C/n$, for some deterministic constant $C>0$.

We first adapt what was done in [KMS21] to prove that there exists a set $\mathcal{E}$ such that $(\hat{W}(n))_{n\geq 0}$ is a stochastic approximation on $\mathcal{E}$.

If we define $\mathfrak{S}_i(G)$ as the number of self-avoiding paths from N_i to F, which we arbitrarily name $c_1^i,\ldots,c^i_{\mathfrak{S}_i(G)}$, then we can define, for every $i\in\{1,2\}$ and j such that $1\leq j\leq c^i_{\mathfrak{S}_i(G)}$:

$$\mathcal{E}^i_j := \left\{ w\in[0,1]^E : \sum_{e\in E: N_i\in e} w_e\geq 1 \text{ and } \forall e\in c^i_j, w_e\geq\frac{1}{\mathfrak{S}_i(G)}\right\}$$

and

$$\mathcal{E} := \mathsf{Conv}\left(\cup_{i=1}^{j} \cup_{j=1}^{\mathfrak{S}_i(G)} \mathcal{E}_j^i\right).$$

We then have the following property, which is the "multi-nest" version of Proposition 2.7 in [KMS21], and can be proven following exactly the same proof:

Proposition 4. *The function F is Lipschitz on the space $\mathcal{E}$. Furthermore the process $\left(\hat{W}(n)\right)_{n\geq 0}$ is a stochastic approximation on $\mathcal{E}$. More precisely,*

$$\hat{W}(n+1) = \hat{W}(n) + \frac{1}{n+2}\left(F(\hat{W}(n)) + \xi(n+1)\right)$$

where for any $e \in \mathcal{E}$, $\xi_e(n+1) := \mathbb{1}_{W_e(n+1)=W_e(n)+1} - p_e(\hat{W}(n))$.

Actually, this proposition is still true if $N(n)$ can have more than 2 values. (Recall that $N(n)$ is the starting nest of the nth ant.)

We can also define the *limiting set*, $L(X) = \cap_{n\geq 0}\overline{\cup_{k\geq n}\{X_k\}}$, and we then have the following proposition, which again is adapted from [KMS21, Corollary 2.6] and originates from [Pem07]:

Proposition 5. *Let X_n be a stochastic approximation, and assume that there exists a deterministic constant $C > 0$ such that $\sup_{n\geq 1} \|\xi_n\| \leq C$ almost surely. We assume moreover that there exist $\mathcal{U}$ and a finite familly $p_1, \ldots, p_k$ such that almost surely $L(X) \subseteq \mathcal{U}$, and for every $w \in \mathcal{U}$, there exists i such that the solution of the ODE $\dot{y} = F(y)$, starting at w, converges to p_i.*

Then, almost surely, there exists $i \in \{1, \ldots, k\}$ such that $L(X) = \{p_i\}$.

The proof of Corollary 2.6 in [KMS21] can easily be adapted to prove this proposition.

3 Proof of Theorem 1

In the whole section, we consider that the graph G is a (ℓ_1, ℓ_2, ℓ_3)-triangle, and we prove Theorem 1. We suppose, without loss of generality, that $\ell_1 \leq \ell_2$.

Summary of the proof. We have seen in Section 2 that our process is a stochastic approximation (see Proposition 4), and thus its asymptotic behaviour is strongly linked with the fixed points of p, i.e. the zeros of F. We use Bendixson–Dulac theorem to prove that a solution of the ODE $\dot{y} = F(y)$ converges (see Lemma 7) and we deduce that the normalised process $(\hat{W}(n))_n$ indeed converges towards some zero of F (see Corollary 8).

Figure 5 illustrates the general behaviour of the ODE $\dot{y} = F(y)$. It highly depends on the parameters (ℓ_1, ℓ_2, ℓ_3 and α_1), but the analysis is the same in all cases, so we illustrate the other cases in Annex A.

Then it remains to determine towards which solution the process converges. We state and prove a series of lemmas (Lemmas 9, 10 and 11) that enable to eliminate the unstable zeros one by one. Figure 5 again helps to have an intuition of the reasoning. For example, we can focus on F_1 and $(\hat{W}_1(n))$. On the figure, the red lines are the zeros of F_1. In the area between the first and the second red lines, $F_1(w) > 0$, and thus, if $y = y(t)$ is such that $\dot{y} = F(y)$, then if $y(t)$ is in this area for some t, its first coordinate will increase. In fact the same happens for $\hat{W}(n)$, and in particular as soon as $\hat{W}_1(n) < \alpha_1$, $F_1(\hat{W}(n)) > 0$, and thus $\hat{W}_1(n)$ tend to increase, regardless of the value of $\hat{W}_3(n)$. Getting rid of the influence of $\hat{W}_3(n)$ enables to study a simpler one-dimensional process, and that is in fact exactly the idea behind the proof of Lemma 9, which states that $\liminf_{n\to\infty} \hat{W}_1(n) \geq \alpha_1$. Lemmas 10 and 11 rely on the same reasonning, even though the technical details are more intricate.

We first prove a very basic lemma that states useful equalities and inequalities on $(W(n))_{n\geq 0}$. Recall that we have set $W_1(n) := W_{a_1}(n)$, $W_2(n) := W_{b_1}(n)$ and $W_3(n) := W_{c_1}(n)$.

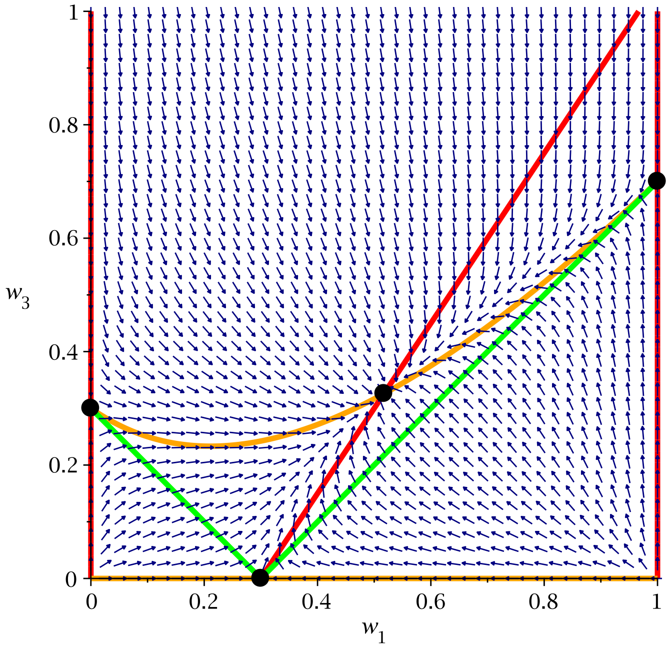


Figure 5: Here $\ell_1 = 2$, $\ell_2 = 4$, $\ell_3 = 3$ and $\alpha_1 = 0.3$.
Blue arrows represent the vector field associated to F. Orange curves represent the solutions to $F_3(w) = 0$, while red ones are solutions to $F_1(w) = 0$. The black dots are thus the solutions to $F(w) = 0$, i.e. the points towards which our process might converge. Recall that those points have coordinates $(\alpha_1, 0)$, $(0, \alpha_1)$, $(1, \alpha_2)$ and (β_1, β_3), with $(\beta_1, \beta_3) \in [0, 1]^2$ if and only if $\ell_2 \leq \ell_1 + \ell_3$ and $\ell_3 \leq \ell_1 + \ell_2$.
The two green lines represent the solutions to $w_3 + w_1 = \alpha_1$ and $w_3 - w_1 = -\alpha_1$, which are relevant since we know (see Remark 1) that asymptotically we can focus on the area where $w_3 + w_1 \geq \alpha_1$ and $w_3 - w_1 \geq -\alpha_1$.
This figure illustrates what happens when $\ell_2 < \ell_1 + \ell_3$ and $\ell_3 < \ell_1 + \ell_2$ (this case is of particular interest since $(\beta_1, \beta_3) \in (0, 1)^2$). Figure 7 in Annex A illustrates the other cases.

Lemma 6. *Almost surely, for every n: $\forall i \leq \ell_1, W_{a_i}(n) = W_1(n)$, $\forall i \leq \ell_2, W_{b_i}(n) = W_2(n)$, and $\forall i \leq \ell_3, W_3(n) = W_{c_1}(n)$. Moreover, almost surely*

$$W_1(n) + W_2(n) = n + 2,$$

$$\liminf_{n\to\infty} \frac{W_1(n) + W_3(n)}{n + 2} \geq \alpha_1 \tag{1}$$

and

$$\liminf_{n\to\infty} \frac{W_2(n) + W_3(n)}{n + 2} \geq \alpha_2 \tag{2}$$

Proof. The equalities on the weights are easy to prove with Remark 1. Moreover, the equality $W_1(n) + W_2(n) = n + 2$ comes from the fact that the nth ant stops after having used either a_{ℓ_1} or b_{ℓ_2}, thus at every step exactly one of these edges is reinforced. We can now focus on the two last inequalities.

Recall that $N(n)$ is the random nest from which the n-th ant starts its walk at step n. Let $i \in \{1, 2\}$. For every n, if $N(n) = N_i$, then either $W_i(n)$ or $W_3(n)$ is increased by 1. Thus we have the following:

$$W_i(n) + W_3(n) \geq 2 + \sum_{k=1}^{n} \mathbb{1}_{N(n)=N_i}.$$

Then we can conclude by using the strong law of large numbers, since $\mathbb{1}_{N(n)=N_i} \overset{iid}{\sim} \mathcal{B}(\alpha_i)$:

$$\liminf_{n\to\infty} \frac{W_i(n) + W_3(n)}{n + 2} \;\geq\; \lim_{n\to\infty} \frac{2 + \sum_{k=1}^{n} \mathbb{1}_{N(n)=N_i}}{n + 2} \;=\; \alpha_i \; a.s.$$

$\square$

First computations We start by computing $p_1(w)$ and $p_3(w)$, using for example a method called *the conductance method*, which is introduced in [LP16, Chapter 2], and using the asumption $w_2 = 1 - w_1$ (which comes from the fact that we have almost surely $\forall n, \hat{W}_1(n) + \hat{W}_2(n) = 1$).

$$\begin{aligned}
p_1(w) &= \sum_{i=1,2} \alpha_i \mathbb{P}\left(W_e(n+1) = W_e(n) + 1 \mid W(n) = w \text{ and } N(n) = N_i\right) \\
&= \alpha_1 \frac{\frac{w_1}{\ell_1}}{\frac{w_1}{\ell_1} + \frac{w_2 w_3}{\ell_2 w_3 + \ell_3 w_2}} + \alpha_2 \frac{\frac{w_1 w_3}{\ell_1 w_3 + \ell_3 w_1}}{\frac{w_1 w_3}{\ell_1 w_3 + \ell_3 w_1} + \frac{w_2}{\ell_2}} \\
&= \frac{w_1\left(\alpha_1 \ell_3 (1-w_1) + \ell_2 w_3\right)}{w_3(\ell_1 + w_1(\ell_2 - \ell_1)) + \ell_3 w_1 (1-w_1)}
\end{aligned}$$

$$\begin{aligned}
p_3(w) &= \alpha_1 \frac{\frac{w_2 w_3}{\ell_2 w_3 + \ell_3 w_2}}{\frac{w_1}{\ell_1} + \frac{w_2 w_3}{\ell_2 w_3 + \ell_3 w_2}} + \alpha_2 \frac{\frac{w_1 w_3}{\ell_1 w_3 + \ell_3 w_1}}{\frac{w_1 w_3}{\ell_1 w_3 + \ell_3 w_1} + \frac{w_2}{\ell_2}} \\
&= \frac{w_3\left(\alpha_1 \ell_1 (1-w_1) + \ell_2 (1-\alpha_1) w_1\right)}{w_3(\ell_1 + w_1(\ell_2 - \ell_1)) + \ell_3 w_1 (1-w_1)}
\end{aligned}$$

We can always write p_1 and p_3 this way, because the denominator is non-zero as soon as $w_1, w_3 \notin \{(0,0),(1,0)\}$, and Lemma 9 enables to see that those points are not even close to the limiting set $L(\hat{W})$. Moreover, we don't need to compute $p_2(w)$, since $p_2(w) = 1 - p_1(w)$.

Computation of the zeros of F. We can compute F_1, F_3, their zeros, and define γ and g, which will be useful later:

$$\begin{aligned}
&F_1(w) = \frac{w_1(1-w_1)\left(\ell_3(\alpha_1 - w_1) + (\ell_2 - \ell_1) w_3\right)}{w_3(\ell_1 + w_1(\ell_2 - \ell_1)) + \ell_3 w_1(1-w_1)} = 0 \\
\iff &w_1 = 0 \text{ or } w_1 = 1 \text{ or } w_3 = \frac{\ell_3(w_1 - \alpha_1)}{\ell_2 - \ell_1} =: \gamma(w_1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&F_3(w) = \frac{w_3\left(\ell_2(1-\alpha_1)w_1 + \ell_1\alpha_1(1-w_1) - \ell_3 w_1(1-w_1) - w_3(\ell_1 + w_1(\ell_2 - \ell_1))\right)}{w_3(\ell_1 + w_1(\ell_2 - \ell_1)) + \ell_3 w_1(1-w_1)} = 0 \\
\iff &w_3 = 0 \text{ or } w_3 = \frac{\ell_2(1-\alpha_1)w_1 + \ell_1\alpha_1(1-w_1) - \ell_3 w_1(1-w_1)}{\ell_1 + w_1(\ell_2 - \ell_1)} =: g(w_1)
\end{aligned}$$

One can check that the only point such that $w_3 = \gamma(w_1)$ and $w_3 = g(w_1)$ is (β_1, β_3) as defined in Theorem 1. Therefore,

$$F(w) = 0 \iff (w_1, w_3) \in \{(0,\alpha_1), (1,\alpha_2), (\alpha_1, 0), (\beta_1, \beta_3)\}. \tag{3}$$

One can then remark that $(\beta_1, \beta_3) \in [0,1]^2$ if and only if $\ell_2 < \ell_1 + \ell_3$ and $\ell_3 < \ell_1 + \ell_2$. In fact, straightforward computation gives:

$$\beta_1 < 1 \iff \ell_2 < \ell_1 + \ell_3 \quad \text{and} \quad \beta_3 > 0 \iff \ell_3 < \ell_1 + \ell_2 \tag{4}$$

(and also $\beta_1 = 1 \iff \ell_2 = \ell_1 + \ell_3$ and $\beta_3 = 0 \iff \ell_3 = \ell_1 + \ell_2$). Moreover, since $\beta_3 = \gamma(\beta_1)$, $\beta_3 \geq 0$ implies directly that $\beta_1 \geq \alpha_1 > 0$. Then, by convexity of g (which can be shown by computing its second derivative), $\beta_3 \leq \max(g(0), g(1)) = \max(\alpha_1, 1-\alpha_1) \leq 1$.

We now prove Lemma 7, which states that solutions to $\dot{y} = F(y)$ starting in the square $[0,1]^2$ converge. This lemma implies (see Corollary 8) that our process $(\hat{W}(n))_n$ almost surely converges.

Lemma 7. *We consider the ODE*

$$\dot{y} = F(y). \tag{5}$$

Then, for any $w \in [0,1]^2$, the solution of the ODE (5) starting at w converges.

Proof. The main idea is to use Bendixson–Dulac method (see for example [HSS98, section 4.1]) to show that such a solution has no orbit, i.e. no nonconstant periodic solution. Then Poincaré-Bendixson theorem [HSS98, theorem 4.1.1] enable to conclude that every solution converges.

First, one can show that if a solution is a periodic orbit, then is stays in $(0,1)\times(0,1]$. In fact, if $\phi : t \mapsto (x(t), y(t))$ is a periodic solution to (5), such that $\exists t : x(t) = 0$, then one can easily show that this implies $x(t) = 0, \forall t$, hence there is a contradiction. Similar reasonning allows to see that for every t, $x(t) \neq 1$ and $y(t) \neq 0$, i.e. $\forall t, \phi(t) \in (0,1)\times(0,1]$.

Now we can focus on proving that there is no periodic orbit on $(0,1)\times(0,1]$.

Let $d := w \mapsto w_3(\ell_1 + w_1(\ell_2 - \ell_1)) + \ell_3 w_1(1-w_1)$, $n_1 := w \mapsto F_1(w)d(w)$, $n_3 := w \mapsto F_3(w)d(w)$, $h : w \mapsto \frac{1}{w_1(1-w_1)w_3}$, and $g := w \mapsto d(w)h(w)$. One can then compute $\nabla_w (Fg)$, by hand or using Maple:

$$\begin{aligned}
\nabla_w (Fg) &= \frac{\partial n_1 h}{\partial w_1}(w) + \frac{\partial n_3 h}{\partial w_3}(w)\\
&= \frac{\partial h}{\partial w_1}(w) n_1(w) + h(w)\frac{\partial n_1}{\partial w_1}(w) + \frac{\partial h}{\partial w_3}(w) n_3(w) + h(w)\frac{\partial n_3}{\partial w_3}(w)\\
&= h(w)\left(\frac{2w_1 - 1}{w_1(1-w_1)} n_1(w) + \frac{\partial n_1}{\partial w_1}(w) + \frac{-1}{w_3} n_3(w) + \frac{\partial n_3}{\partial w_3}(w)\right)\\
&= \left(\ell_3 w_1^2 + ((\ell_1 - \ell_2)w_3 - \ell_3)w_1 - w_3\ell_1\right) h(w)\\
&= g(w)
\end{aligned}$$

Then, for all w_1, w_3, we have $d(w) = \ell_1 w_3(1-w_1) + \ell_2 w_3 w_1 + \ell_3 w_1(1-w_1) > 0$ on $(0,1)\times(0,1]$.

We have found g a $\mathcal{C}^1$ function such that $\nabla_w (Fg) > 0$ on $(0,1)\times(0,1]$, therefore we can apply Bendixson-Dulac theorem [HSS98, Section 4.1]: every solution to (5) starting in $(0,1)\times(0,1]$ has no orbit. And therefore it converges, thanks to Poincaré-Bendixson theorem [HSS98, theorem 4.1.1] (if it was not converging, then Poincaré-Bendixson theorem would imply the existence of a periodic orbit, and thus raise a contradiction). $\square$

We thus have the following corollary, which states that the normalised process $(\hat{W}(n))_n$ almost surely converges.

Corollary 8. *Almost surely,* $\lim_{n\to\infty} \hat{W}(n)$ *exists, and*

$$\lim_{n\to\infty} \hat{W}(n) \in \{(1,\alpha_2), (\alpha_1, 0), (\beta_1, \beta_3)\}.$$

Proof. Recall that $\{w : F(w) = 0\} = \{(1,\alpha_2), (0,\alpha_1), (\alpha_1,0), (\beta_1,\beta_3)\}$. Then as a direct corollary of Lemma 7 and Proposition 5 (which can be applied to $\left(\hat{W}(n)\right)_n$ since Proposition 4 states that it is a stochastic approximation), we have that almost surely

$$\lim_{n\to\infty} \hat{W}(n) \in \{(1,\alpha_2), (0,\alpha_1), (\alpha_1,0), (\beta_1,\beta_3)\}.$$

Combining this with Lemma 9, we get that almost surely,

$$\hat{W}(n) \underset{n\to\infty}{\longrightarrow} (\beta_1,\beta_3),\ \hat{W}(n) \underset{n\to\infty}{\longrightarrow} (1,\alpha_2) \quad \text{or} \quad \hat{W}(n) \underset{n\to\infty}{\longrightarrow} (\alpha_1, 0). \tag{6}$$

$\square$

We now eliminate the unstables zeros one by one, starting by $(0,\alpha_1)$ with the following lemma.

Lemma 9. *Recall that* $\ell_1 \leq \ell_2$*. Then, almost surely:*

$$\liminf_{n\to\infty} \frac{W_1(n)}{n} \geq \alpha_1.$$

Proof. We prove this lemma by building a function g such that a g-urn process converges (we show this using Proposition 2), and such that our process $(W(n))_n$ dominates a g-urn process.

We consider $f : w_3 \mapsto p_1(w)$. Then, one can compute

$$f'(w_3) = \frac{\ell_3 w_1(1-w_1)\left(\ell_2(1-\alpha_1)w_1 - \ell_1\alpha_1(1-w_1)\right)}{\left(w_3(\ell_1 + w_1(\ell_2-\ell_1)) + \ell_3 w_1(1-w_1)\right)^2}.$$

f' has constant sign, thus f is monotonous.
Moreover, $f' > 0$ if and only if $w_1 \geq \frac{\ell_1\alpha_1}{\ell_2(1-\alpha_1)+\ell_1\alpha_1} =: c_1$.
We can then consider $g : w_1 \mapsto \inf_{w_3} p_1(w)$. g is such that:

$$g(w_1) = \begin{cases} p_1(w_1, 0) = \alpha_1 & \text{if } w_1 \geq c_1, \\ p_1(w_1, 1) = \frac{w_1(\alpha_1\ell_3(1-w_1)+\ell_2)}{\ell_1(1-w_1)+\ell_2 w_1+\ell_3 w_1(1-w_1)} & \text{otherwise,} \end{cases}$$

and g is continuous, since $p_1(c_1, 1) = \alpha_1$. Moreover, $\ell_2 \geq \ell_1$ implies that $\alpha_1 \geq c_1$, and then a little computation enables to prove that $g(w_1) = w_1$ if and only if $w_1 = 0$ or $w_1 = \alpha_1$. Finally, $g(w_1) \underset{w_1 \to 0}{\sim} w_1 \frac{\alpha_1\ell_3+\ell_2}{\ell_1} > 0$, so it follows from Proposition 5 that any g-urn process converges to the only stable fixed-point of g, i.e. α_1.

To conclude, $\left(\frac{W_1(n)}{n}\right)_n$ stochastically dominates a g-urn process, so in particular

$$\liminf_{n\to\infty} \frac{W_1(n)}{n} \geq \alpha_1.$$

□

The following lemma enables to show that, when $\ell_3 < \ell_1 + \ell_2$, $\hat{W}(n)$ does not tend to $(\alpha_1, 0)$.

Lemma 10. *If $\ell_3 < \ell_1 + \ell_2$, then there exists $c > 0$ such that a.s.*

$$\liminf_{n\to\infty} \frac{W_3(n)}{n} \geq c$$

Proof. We prove that there exists a constant $c > 0$ such that for every $\varepsilon_1 > 0$,

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{W_3(n)}{n} \geq c\right) \geq 1 - \varepsilon_1.$$

Our purpose now is to define an event $\mathcal{A}_{\varepsilon_1}$ such that $\mathbb{P}(\mathcal{A}_{\varepsilon_1}) \geq 1 - \varepsilon_1$ and conditionally on $\mathcal{A}_{\varepsilon_1}$, we can do as in the previous lemma, i.e. show that our process $(W(n))_n$ dominates some g-urn process which almost surely has a positive liminf. More precisely, we will prove that there exist $\varepsilon > 0$ and $c > 0$ such that, conditionally on $\mathcal{A}_{\varepsilon_1}$, for n large enough, $\hat{W}_3(n) \leq c$ implies $p_3(W_1(n), W_3(n)) \geq (1+\varepsilon)W_3(n)$, and then we will be able to conclude with [KMS21, Corollary 2.2]. The event $\mathcal{A}_{\varepsilon_1}$ will be built in order to make use of Equations (1) and (2) in Lemma 6. We need to use this lemma, because in some cases (depending on α_1, ℓ_1, ℓ_2 and ℓ_3), for every c there exist some (w_1, w_3) such that $w_3 < c$ and $p_3(w_1, w_3) \leq w_3$. Figure 2 illustrates the fact that, even in those cases, Lemma 6 enables to focus on the pairs (w_1, w_3) verifying $w_3 + w_1 \geq \alpha_1 - \varepsilon_2$ and $w_3 + \alpha_1 \geq w_1 - \varepsilon_2$, and on this set of pairs we don't have this issue.

Denoting that $w_3(\ell_1 + w_1(\ell_2 - \ell_1)) + \ell_3 w_1(1-w_1) > 0$, we do the following computation, and define g_ε:

$$\begin{aligned}
& p_3(w) = \frac{w_3\left(\alpha_1\ell_1(1-w_1) + \ell_2(1-\alpha_1)w_1\right)}{w_3(\ell_1 + w_1(\ell_2-\ell_1)) + \ell_3 w_1(1-w_1)} \;\geq\; (1+\varepsilon)\,w_3 \\
\iff\; & \alpha_1\ell_1(1-w_1) + \ell_2(1-\alpha_1)w_1 \;\geq\; (1+\varepsilon)\left(w_3(\ell_1 + w_1(\ell_2-\ell_1)) + \ell_3 w_1(1-w_1)\right) \\
\iff\; & (1+\varepsilon)\left(\ell_1 + w_1(\ell_2-\ell_1)\right) w_3 \;\leq\; \alpha_1\ell_1(1-w_1) + \ell_2(1-\alpha_1)w_1 - (1+\varepsilon)\,\ell_3 w_1(1-w_1) \\
\iff\; & w_3 \;\leq\; \frac{\alpha_1\ell_1(1-w_1) + \ell_2(1-\alpha_1)w_1 - (1+\varepsilon)\,\ell_3 w_1(1-w_1)}{(1+\varepsilon)\left(\ell_1 + w_1(\ell_2-\ell_1)\right)} =: g_\varepsilon(w_1).
\end{aligned}$$

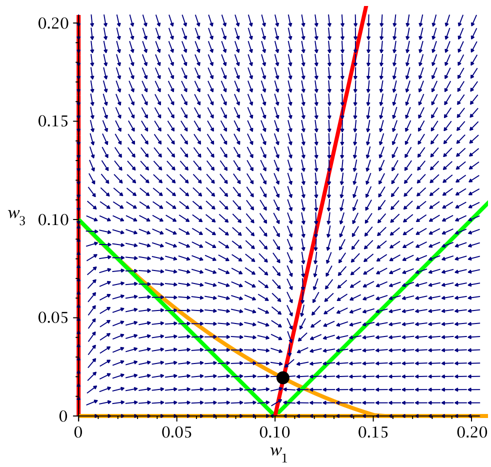


Figure 6: Here $\ell_1 = 8$, $\ell_2 = 12$, $\ell_3 = 18$ and $\alpha_1 = 0.1$. The black dot corresponds to the point (β_1, β_3). We use the same colour coding as in Figure 5.
(β_1, β_3) is the only point belonging to both the orange curve (representing $g(w_1)$) and the red curve, and $\beta_3 > 0$, thus the orange curve can't cross the green one on a point with a zero ordinate. Thus $\inf_{w_1: g(w_1)+w_1 \geq \alpha_1, g(w_1)+\alpha_1 \geq w_1} g(w_1) > 0$.

Now we want to prove that there exists $\varepsilon > 0$ such that $\inf_{w_1:(w_1, g_\varepsilon(w_1)) \in \mathcal{E}'} g_\varepsilon(w_1) > 0$. Intuitively, g_ε is close to g for ε small enough, so we can rely on the fact that this is true for g. Figure 6 illustrates the intuition of why it is always true for g, even if g might be negative on a subset of $[0, 1]$. We now prove it for g_ε.

Straightforward computation (exactly the same as the one for (β_1, β_3)) shows that there is a unique solution to $g_\varepsilon(w_1) = \gamma(w_1)$; let $\beta_1^{(\varepsilon)}$ be this solution, and let $\beta_3^{(\varepsilon)} := \gamma(w_1)$. $\beta_1^{(\varepsilon)}$ is such that:

$$\beta_1^{(\varepsilon)} = \frac{\alpha_1 \ell_1 \left((1+\varepsilon)\ell_3 + \ell_2 - \ell_1\right)}{(1+\varepsilon)\ell_1\ell_3 + (\ell_2 - \ell_1)\left((1-\alpha_1)((1+\varepsilon)\ell_3 - \ell_2) + \alpha_1\ell_1\right)}.$$

It is quite obvious that $\beta_1^{(\varepsilon)} \underset{\varepsilon \to 0}{\to} \beta_1$ and $\beta_3^{(\varepsilon)} \underset{\varepsilon \to 0}{\to} \beta_3$. So there exists ε such that $\beta_3^{(\varepsilon)} > 0$ (since we have seen that $\beta_3 > 0 \iff \ell_3 < \ell_1 + \ell_2$). Here one can notice that we don't need to have $(\beta_1, \beta_3) \in [0, 1]^2$, the proof still works if $\beta_1 > 1$.

We prove by contradiction that $\inf_{w_1:(w_1, g_\varepsilon(w_1)) \in \mathcal{E}'} g_\varepsilon(w_1) > 0$. If $\inf_{w_1:(w_1, g_\varepsilon(w_1)) \in \mathcal{E}'} g_\varepsilon(w_1) = 0$, then there would exist a sequence $(w_1(n))_n$ such that for every n, $(w_1(n), g_\varepsilon(w_1(n))) \in \mathcal{E}'$, and $g_\varepsilon(w_1(n)) \underset{n \to \infty}{\to} 0$. Since $(w_1(n), g_\varepsilon(w_1(n))) \in \mathcal{E}'$, $\alpha_1 + g_\varepsilon(w_1(n)) \geq w_1(n) \geq \alpha_1 - g_\varepsilon(w_1(n))$, and therefore we can deduce that $w_1(n) \underset{n \to \infty}{\to} \alpha_1$.

But then, by continuity of g_ε, we would have $g_\varepsilon(w_1(n)) \underset{n \to \infty}{\to} g_\varepsilon(\alpha_1) = 0$. Since $\gamma(\alpha_1) = 0$, and since $\beta_1^{(\varepsilon)}$ is the unique solution to the equation $g_\varepsilon(w_1) = \gamma(w_1)$, it implies that $\beta_1^{(\varepsilon)} = \alpha_1$ and $\beta_3^{(\varepsilon)} = 0$, which is a contradiction, by definition of ε.

So we have proven that

$$\inf_{w_1:(w_1, g_\varepsilon(w_1)) \in \mathcal{E}'} g_\varepsilon(w_1) > 0.$$

We can prove the same result on the set

$$\mathcal{E}'_{\varepsilon_2} = \{w \in \mathcal{E} : w_1 + w_2 = 1, w_3 + w_1 \geq \alpha_1 - \varepsilon_2, \text{ and } w_3 + \alpha_1 \geq w_1 - \varepsilon_2\}$$

as soon as ε_2 is small enough. In fact, g_ε is continuous and $\{w_1 : (w_1, g_\varepsilon(w_1)) \in \mathcal{E}'_{\varepsilon_2}\}$ is an interval that tends to $\{w_1 : (w_1, g_\varepsilon(w_1)) \in \mathcal{E}'\}$, therefore

$$\inf_{w_1:(w_1, g_\varepsilon(w_1)) \in \mathcal{E}'_{\varepsilon_2}} g_\varepsilon(w_1) \underset{\varepsilon_2 \to 0}{\longrightarrow} \inf_{w_1:(w_1, g_\varepsilon(w_1)) \in \mathcal{E}'} g_\varepsilon(w_1) > 0,$$

Thus there exists ε_2 such that $c := \inf_{w_1:(w_1, g_\varepsilon(w_1)) \in \mathcal{E}'_{\varepsilon_2}} g_\varepsilon(w_1) > 0$.

The idea now is that, even though it is possible to have $\inf_{w_1\in[\alpha_1,1]} g_\varepsilon(w_1) \le 0$, we can use Lemma 6 to say that almost surely we can ignore what happens outside $\mathcal{E}'_{\varepsilon_2}$ after a certain time (i.e. a certain number of steps), and thus after that time we can apply Corollary 3.

Let $\varepsilon_1 > 0$. As direct corollary of Lemma 6, there exists n_{ε_1} such that

$$\mathbb{P}\left(\forall n \ge n_{\varepsilon_1}, \frac{W_1(n)+W_3(n)}{n+2} \ge \alpha_1 - \varepsilon_2 \text{ and } \frac{W_3(n)-W_1(n)}{n+2} \ge -\alpha_1 - \varepsilon_2\right) \ge 1-\varepsilon_1$$

i.e., if we define the event $\mathcal{A}_{\varepsilon_1} := \left\{\forall n \ge n_{\varepsilon_1}, \left(\frac{W_1(n)}{n+2}, \frac{W_3(n)}{n+2}\right) \in \mathcal{E}'_{\varepsilon_2}\right\}$, then $\mathbb{P}\left(\mathcal{A}_{\varepsilon_1}\right) \ge 1-\varepsilon_1$.

Thus, if we define $G : w_3 \mapsto \inf_{w_1:(w_1,w_3)\in\mathcal{E}'_{\varepsilon_2}} p_3(w_1,w_3)$, then G is such that, conditionally on $\mathcal{A}_{\varepsilon_1}$, for every $n \ge n_{\varepsilon_1}$,

$$\mathbb{P}\left(W_3(n+1) = W_3(n)+1 | W(n)\right) \ge G(\hat{W}_3(n)).$$

Moreover, thanks to the computation we did on g_ε, we know that $G(x) \ge (1+\varepsilon)x$, for every $x < c$. Finally, Corollary 3 conditionally on the event $\mathcal{A}_{\varepsilon_1}$ enables to conclude:

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{W_3(n)}{n} \ge c\right) \ge 1-\varepsilon_1.$$

Since this is true for every ε_1, we can conclude that almost surely

$$\liminf_{n\to\infty} \frac{W_3(n)}{n} \ge c.$$

□

This last lemma enables to show that, when $\ell_2 < \ell_1 + \ell_3$, $\hat{W}(n)$ does not converge to $(1, \alpha_2)$.

Lemma 11. *If $\ell_2 < \ell_1 + \ell_3$, there exists $c < 1$ such that*

$$\limsup_{n\to\infty} \hat{W}_1(n) \le c.$$

Proof. Here again, the proof relies on the coupling of $\hat{W}_1(n)$ with a one-dimentional G-urn process for some G on which we can apply Corollary 3.

We can deduce from Corollary 8 that almost surely: $\limsup_{n\to\infty} \hat{W}_3(n) \le \max\left(\beta_3, \alpha_2\right)$.

If we let $c_0 := \max\left(\beta_3, \alpha_2\right)$, and $\varepsilon_0 > 0$ be such that $c_0 + \varepsilon_0 < \gamma(1)$ (this is possible since $\gamma(1) = \frac{\ell_3\alpha_2}{\ell_2-\ell_1}$ and $\ell_3 > \ell_2 - \ell_1$, so $\alpha_2 < \gamma(1)$, and $\beta_1 < 1$ so $\beta_3 = \gamma(\beta_1) < \gamma(1)$), then for every ε there exists n_ε such that $\mathbb{P}\left(\forall n \ge n_\varepsilon, \hat{W}_3(n) \le c_0 + \varepsilon_0\right) \ge 1-\varepsilon$. We define the event $\mathcal{A}_\varepsilon := \left\{\forall n \ge n_\varepsilon, \hat{W}_3(n) \le c_0 + \varepsilon_0\right\}$.

We have seen, in the proof of Lemma 9, that $w_3 \mapsto p_1(w)$ is monotonous, thus we have the following:

$$\sup_{w_3\in[0,c_0+\varepsilon_0]} p_1(w) = \max\left(p_1(w_1,0), p_1(w_1, c_0+\varepsilon_0)\right).$$

We want to prove that there exist $\varepsilon_1 > 0$ and $c < 1$ such that for every $w_1 \in (c, 1)$,

$$\sup_{w_3\in[0,c_0+\varepsilon_0]} p_1(w) \le (1-\varepsilon)w_1. \tag{7}$$

For every $\varepsilon_1 \in (0, \alpha_2)$, $p_1(w_1, 0) = \alpha_1 < (1-\varepsilon_1)$ and $\gamma\left(w_1\right) \underset{w_1\to 1}{\longrightarrow} \gamma(1) > c_0 + \varepsilon_0$, thus there exists c_{ε_1} such that

$$\forall w_1 > c_{\varepsilon_1},\ p_1(w_1, 0) = \alpha_1 < (1-\varepsilon_1)w_1 \quad \text{and} \quad \gamma(w_1) > c_0 + \varepsilon_0.$$

In addition to that, we can assume that $c_{\varepsilon_1} > \alpha_1$ and $\varepsilon_1 \mapsto c_{\varepsilon_1}$ is decreasing. Then, one can check that $\gamma(w_1) > c_0 + \varepsilon_0$ is equivalent to

$$\alpha_1\ell_3(1-w_1) + \ell_2(c_0+\varepsilon_0) < (c_0+\varepsilon_0)(\ell_1 + w_1(\ell_2-\ell_1)) + \ell_3 w_1(1-w_1)$$

And this finally proves that there exists ε_1 verifying

$$\alpha_1\ell_3(1-w_1)+\ell_2(\alpha_2+\varepsilon_0)\leq(1-\varepsilon_1)\left((\alpha_2+\varepsilon_0)(\ell_1+w_1(\ell_2-\ell_1))+\ell_3w_1(1-w_1)\right). \quad (8)$$

One can then verify that the inequality (8) is equivalent to $p_1(w_1,\alpha_2+\varepsilon_0)\leq(1-\varepsilon_1)w_1$, for any $w_1\in(0,1)$. We can deduce that (7) is true.

(7) implies the existence of $G:[0,1]\to[0,1]$ such that for every $x\in(c_{\varepsilon_1},1)$, $G(x)<(1-\varepsilon_1)x$ and, conditionally on the event $\mathcal{A}_\varepsilon$,

$$\forall n\geq n_\varepsilon,\ \mathbb{P}\left(W_1(n+1)=W_1(n)+1\mid W(n)\right)\ \leq\ G(W_1(n)).$$

It then follows from Corollary 3 that, conditionally on $\mathcal{A}_\varepsilon$, $\limsup_{n\to\infty}\hat{W}_1(n)\leq c_{\varepsilon_1}$, i.e. $\mathbb{P}\left(\limsup_{n\to\infty}\hat{W}_1(n)\leq c_{\varepsilon_1}\right)\geq 1-\varepsilon$. This is true for every $\varepsilon>0$, so almost surely

$$\limsup_{n\to\infty}\hat{W}_1(n)\leq c_{\varepsilon_1}.$$

□

Now we have proven enough intermediate results to conclude the proof of Theorem 1.

Proof of Theorem 1. We have seen that we have three distinct cases:

- If $\ell_3\geq\ell_1+\ell_2$: In this case, $(\beta_1,\beta_3)\notin[0,1]^2$ (except if $\ell_3=\ell_1+\ell_2$, but then $(\beta_1,\beta_3)=(\alpha_1,0)$).

  Lemma 11 implies that $\mathbb{P}\left(\hat{W}(n)\underset{n\to\infty}{\longrightarrow}(1,\alpha_2)\right)=0$, and together with Corollary 8, we finally get that almost surely,
  $$\hat{W}(n)\underset{n\to\infty}{\longrightarrow}(\alpha_1,0).$$

- If $\ell_2\geq\ell_1+\ell_3$: Here again, $(\beta_1,\beta_3)\notin[0,1]^2$ (except if $\ell_2=\ell_1+\ell_3$, but then $(\beta_1,\beta_3)=(1,\alpha_2)$).

  Then Lemma 10 gives $\mathbb{P}\left(\hat{W}(n)\underset{n\to\infty}{\longrightarrow}(0,\alpha_1)\right)=0$, and combining this with Corollary 8 gives that almost surely,
  $$\hat{W}(n)\underset{n\to\infty}{\longrightarrow}(1,\alpha_2).$$

- Finally, if $\ell_2<\ell_1+\ell_3$ and $\ell_3<\ell_1+\ell_2$, the same reasonning combining Lemma 10, Lemma 11 and Corollary 8 enables to prove that almost surely,

$$\hat{W}(n)\underset{n\to\infty}{\longrightarrow}(\beta_1,\beta_3).$$

□

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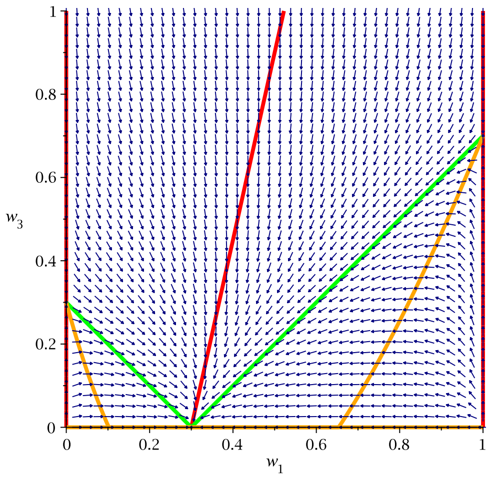
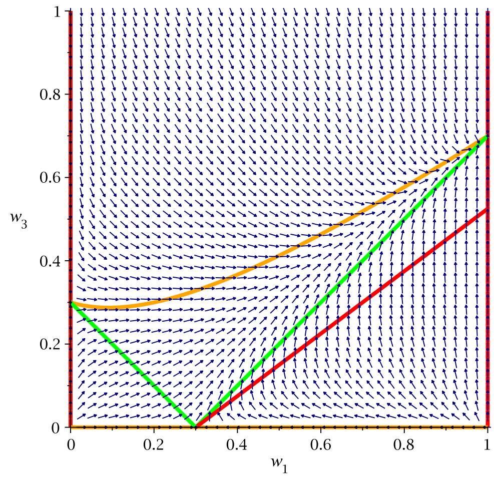
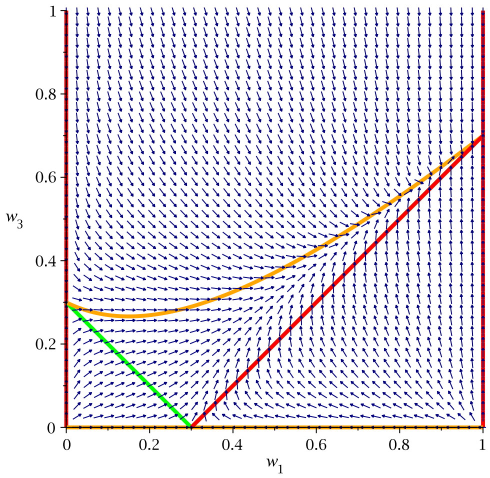
A Figures illustrating the other cases



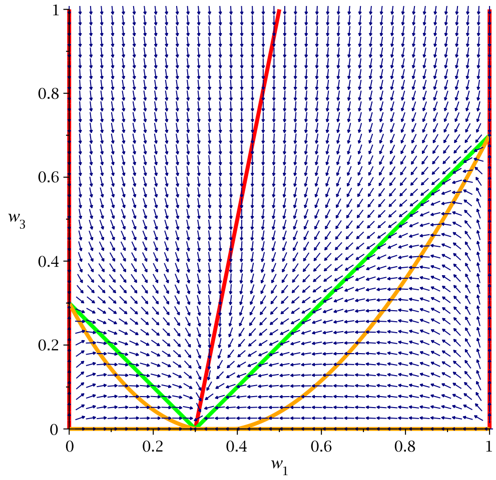
(a) Here $\ell_1 = 2$, $\ell_2 = 4$, $\ell_3 = 9$ and $\alpha_1 = 0.3$. This figure illustrates what happens when $\ell_3 > \ell_1 + \ell_2$.



(b) Here $\ell_1 = 2$, $\ell_2 = 6$, $\ell_3 = 3$ and $\alpha_1 = 0.3$. This figure illustrates what happens when $\ell_2 > \ell_1 + \ell_3$.



(c) Here $\ell_1 = 2$, $\ell_2 = 5$, $\ell_3 = 3$ and $\alpha_1 = 0.3$. This figure illustrates what happens when $\ell_2 = \ell_1 + \ell_3$ (i.e. $(\beta_1, \beta_3) = (1, \alpha_2)$).



(d) Here $\ell_1 = 2$, $\ell_2 = 3$, $\ell_3 = 5$ and $\alpha_1 = 0.3$. This figure illustrates what happens when $\ell_3 = \ell_1 + \ell_2$ (i.e. $(\beta_1, \beta_3) = (\alpha_1, 0)$).

Figure 7: This figure represents the same objects as Figure 5 (see its caption for a detailled explanation), but in different cases that highlight really distinct behaviors according to the parameters ℓ_1, ℓ_2 and ℓ_3.