Towards Topological Classification of Univariate Complex Polynomials

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Abstract. The study of the topological classification of complex polynomials began in the XIX-th century by Luroth (1871), Clebsch (1873) and Hurwitz (1891). However, very few things are known today about the problem. A vast computer experiment allowed us to achieve a classification of all the polynomials up to degree 11 (until 1996, such a classification was known up to degree 6). These new data change entirely the global view of the problem and permit to formulate some plausible conjectures that may eventually lead to a solution of the problem.

1 Introduction

1.1 Statement of the Problem

Let $P_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and $P_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be two polynomials of degree n, where $\overline{\mathbb{C}} \simeq S^2$ is the Riemann complex sphere. We say that P_1 and P_2 are *topologically equivalent* if there exist two orientation preserving homeomorphisms $h_1 : S^2 \to S^2$ and $h_2 : S^2 \to S^2$ such that the following diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{h_1} & S^2 \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle P_2} \\ S^2 & \xrightarrow{h_2} & S^2 \end{array}$$

is commutative. (We would like to attract the reader's attention to the possibility of performing a homeomorphism h_2 on the lower level. Many authors consider a triangular diagram, in which only h_1 is present, while h_2 is replaced by the identity, and also call the corresponding problem the problem of a "topological classification". These two problems are very different.)

The study of polynomials and, more generally, branched covers of S^2, up to topological equivalence began in the XIX-th century by Luroth [19], Clebsch [4] and Hurwitz [14]. In the XX-th century the problem reappeared after a series of short notes [6], [25], [1] and [27]. In Sect. 1.4 we also mention several subsequent publications, but our list is certainly incomplete. We just note that the fundamental theorem of S. Zdravkovska [27] lays a foundation for our study.

1.2 Basic Notions

A *critical value* of a polynomial P, or a *branch point* of the corresponding ramified covering of S^2, is a point $y \in \overline{\mathbb{C}}$ such that the equation $P(x) = y$ has multiple roots. (The point $y = \infty$ must also be considered as a critical value; the corresponding "solution" $x = \infty$ has multiplicity $n = \deg P$.) Finite critical values of P are the images under P of the roots of its derivative P'. Let $y_1, \dots, y_k$ be all the finite critical values of a polynomial P, $\deg P = n$, and let y_0 be a base-point different from all the y_i. Choosing appropriate loops starting and ending at y_0 and going around the branch points, we may associate to the polynomial P a sequence of permutations $[g_1, \dots, g_k]$, $g_i \in S_n$. The permutations g_i act on the n points of the preimage $P^{-1}(y_0)$. They generate the permutation group

$$G = <g_1, \dots, g_k> \leq S_n \ , \tag{1}$$

the *monodromy group* of the ramified covering corresponding to P.

Let $\lambda^{(i)} \vdash n$, $\lambda^{(i)} = (d_{i1}, d_{i2}, \dots, d_{ip_i})$, $\sum_{j=1}^{p_i} d_{ij} = n$ be the cyclic structure of the permutation g_i, $i = 1, \dots, k$, or, equivalently, the multiplicity partition for the roots of the equation $P(x) = y_i$. The permutations g_i must satisfy the following conditions:

1. The product $g_1 g_2 \dots g_k$ is a long cycle: the points in $P^{-1}(y_0)$ may be numbered in such a way that

   $$g_1 g_2 \dots g_k = c := (1, 2, \dots, n) \ . \tag{2}$$

   The reason for that is that the loop going around all the points y_i may also be considered as a loop going around the infinity.
2. Planarity condition: the total number of cycles in the permutations $g_1, \dots, g_k$ is equal to $\sum_{i=1}^{k} p_i = (k-1)n + 1$, or, equivalently,

   $$\sum_{i=1}^{k} \sum_{j=1}^{p_i} (d_{ij} - 1) = n - 1 \ . \tag{3}$$

   This equality follows from the Riemann–Hurwitz formula. It also means that $\deg P' = n-1$ (the roots of the derivative P' have the multiplicities $(d_{ij}-1)$).

Definition 1. A sequence of permutations $[g_1, \dots, g_k]$ satisfying the conditions (2) and (3) is called a (rooted) *cactus*. A sequence of partitions $[\lambda^{(1)}, \dots, \lambda^{(k)}]$ of number n satisfying the condition (3) is called a *polynomial passport*.

The choice of the term "cactus" is motivated by a possibility to represent this object graphically: see [11] and [9]. The term "passport" was proposed by Protopopov [22].

Theorem 2. (A) *For an arbitrary choice of $y_1, \dots, y_k \in \mathbb{C}$, and for an arbitrary cactus $[g_1, \dots, g_k]$, there exists a polynomial P with the critical values y_i and the corresponding monodromy generators g_i. This polynomial is unique up to an affine change of the variable x.*

(B) *For any polynomial passport $[\lambda^{(1)}, \dots, \lambda^{(k)}]$ there exists at least one cactus $[g_1, \dots, g_k]$ such that $\lambda^{(i)}$ is the cyclic structure of g_i.*

The part (A) is a particular case of the Riemann's existence theorem; for its proof see [26]. A simple though important statement (B) was proved several times, first in [25], then in [8], [16] and [9]. Indirectly it is also proved in [11], since the enumerative formula for cacti (6) always gives a positive number.

1.3 Braid Group Action

Let an operation σ_i $(i = 1, \dots, k-1)$ act on a cactus $[g_1, \dots, g_k]$ in the following way:

$$\sigma_i(g_i) = g_{i+1}, \quad \sigma_i(g_{i+1}) = g_{i+1}^{-1} g_i g_{i+1}, \quad \text{and} \quad \sigma_i(g_j) = g_j \text{ for } j \neq i \ . \qquad (4)$$

This action obviously preserves the product $g_1 \dots g_k$. The following theorem of S. Zdravkovska [27] is fundamental:

Theorem 3. (A) *The operations σ_i, $i = 1, \dots, k-1$ generate an action of the braid group B_k. That is, they satisfy the relations*

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2, \quad \text{and} \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ . \qquad (5)$$

(B) *The classes of the topological equivalence of complex polynomials are in one to one correspondence with the orbits of this action.*

The operations σ_i were introduced by Hurwitz in 1891 [14]. However, he did not notice that they generate an action of a group, and the braid group B_k was introduced only 34 years later [2]. Hurwitz also proved the "if" part of the statement (B). The proof of the "only if" part is based on a theorem of Kneser [18] (1926) which states that the topological space of the homeomorphisms of S^2 is path connected.

Invariants of the Action. There exist the following obvious invariants of the braid group action:

(1) the monodromy group $G = <g_1, \dots, g_k>$;

(2) the set of the conjugacy classes of the permutations $g_1, \dots, g_k$;

(3) the non-ordered passport.

Note also that the case $k = 2$ is trivial (only from the point of view of the braid group action; otherwise it is very interesting, see [24]): we have $\sigma_1^2(g_1) = g_1^c$ and $\sigma_1^2(g_2) = g_2^c$, where $c = g_1 g_2 = (1, \dots, n)$. In what follows we study only the case $k \geq 3$.

1.4 What Was Known Before

Here we briefly mention the results concerning the problem of the topological classification of polynomials known before our experimental studies.

(1) Let us call a critical value *simple* if the corresponding partition is 21^{n-2}. The classics of the XIX-th century have shown that rational or meromorphic

functions with only simple critical values constitute a single class of the topological equivalence. This result implies nothing at all for the polynomials, as every polynomial has a non-simple critical value at infinity.

(2) P. Deligne [7] proved that *generic* polynomials, that is, polynomials with $k = n-1$ simple finite critical values, constitute a single class of the topological equivalence.[1]

(3) The previous result is covered by a theorem of S. Zdravkovska [27] which states that if for every critical value there is only one critical point (that is, if all the partitions in the passport are of the form $m1^{n-m}$, $m \geq 2$), then there exists a single class of the topological equivalence of polynomials.

(4) S. Natanzon [21] proved that if all finite critical values except one are simple, then the corresponding class is unique.

(5) The strongest result that covers almost all the previous ones (except (3)) belongs to A. Khovanskii and S. Zdravkovska [16]. For a polynomial passport, let us call its *defect* the number $D = \sum_{d_{ij}>1} d_{ij}$, where *the sum is taken over all non-simple critical values*. Then the following holds:

Theorem 4. *If $D \leq n+1$, then the corresponding class of the topological equivalence of polynomials is unique.*

Simple arithmetic considerations show that the maximal value of k, for which there exist polynomial passports with $D > n+1$, is equal to $k_{\max} = [3(n-2)/4]$, where the square brackets denote the integer part.

2 Experimental Study

2.1 Enumerative Formula

Suppose you have taken a polynomial passport, you have chosen a cactus having this passport, and you have computed its orbit under the braid group action; suppose that the orbit turned out to be of size, say, 12000. So what? This information alone does not allow you to conclude if there are other orbits corresponding to the same passport or not. The following enumerative formula due to Goulden and Jackson [11] is very helpful, as it permits us to know in advance the total number of cacti with a given passport. If, for example, this total number is 12000, then everything is OK: we may affirm that there is a single orbit.

For a given partition $\lambda \vdash n$, $\lambda = (d_1, \ldots, d_p) = 1^{p_1} 2^{p_2} \ldots n^{p_n}$ (where $\sum p_i = p$ and $\sum i p_i = n$), let us denote

$$N(\lambda) = \frac{(p-1)!}{p_1! p_2! \ldots p_n!} \ .$$

Theorem 5. *The number of rooted cacti having the passport $[\lambda^{(1)}, \ldots, \lambda^{(k)}]$ is equal to*

$$n^{k-1} \prod_{i=1}^{k} N(\lambda^{(i)}) \ . \tag{6}$$

[1] This work remained unpublished, and we have never seen it; here we quote [17].

2.2 About the Algorithm

Algorithmic aspects of computing the orbits of an action of a group constitute a subject interesting in itself. A detailed description of our algorithm will be presented elsewhere. Here we only give a few remarks concerning some particular features of the algorithm.

(1) A *non-rooted cactus* is an equivalence class of the rooted cacti under the conjugation by the long cycle $c = (1, 2, \ldots, n)$. The orbit we compute consists of non-rooted cacti rather than of rooted ones. The results thus obtained are equivalent, as the braid $(\sigma_1 \ldots \sigma_{k-1})^k$ transforms a cactus $[g_1, \ldots, g_k]$ into the cactus $[g_1^c, \ldots, g_k^c]$. Taking non-rooted cacti as representatives permits us to "almost" divide the orbit size by n. More exactly, the automorphism group of a non-rooted cactus is a cyclic group generated by c^m for some m; we say that it has a symmetry of order n/m. If this group is $\{\mathrm{id}\}$ (asymmetric case), to each non-rooted cactus there correspond n rooted ones. If it is non-trivial (symmetric case), to a non-rooted cactus there correspond m rooted ones. But anyway, symmetric and asymmetric cacti lie in different orbits, because the monodromy group of a symmetric cactus possesses a non-trivial centralizer, while for the asymmetric one it does not.

(2) The operations σ_i permute partitions $\lambda^{(i)}$ and $\lambda^{(i+1)}$ in the passport. Therefore, we must put into the same orbit all the cacti corresponding to all the ordered passports that may be obtained from the given passport by permuting its partitions. This makes the orbit much bigger than we would like it to be.

If we "forget" the overcrossings and undercrossings of the braid strings, or, algebraically, if we add the relations $\sigma_i^2 = \mathrm{id}$ to that of (5), we obtain a group homomorphism that projects the braid group B_k onto the symmetric group S_k. The *pure braid group* is the subgroup $P_k < B_k$ which is the preimage, under this homomorphism, of the trivial subgroup $\{\mathrm{id}\}$ of S_k. Now, let $\mu \vdash k$, $\mu = (m_1, \ldots, m_r)$ be a partition of k. Let us call the *Young braid group*, and denote by Y_μ, the subgroup of B_k which is the preimage, under the same homomorphism, of the Young subgroup $S_\mu = S_{m_1} \times \ldots \times S_{m_r} \leq S_k$. The group Y_μ is the biggest subgroup of B_k that preserves the *ordered* passport in which the first m_1 partitions are equal; the next m_2 ones are equal (but different from the first m_1 ones), etc. The orbits we compute are in fact the orbits of the action of the Young braid group Y_μ, and not that of B_k.

The pure braid group P_k is no good for this purpose, as it may create an artificial splitting of orbits. For example, for the passport $[2^5, 2 \times 2^2 1^6]$ there are 4 orbits of the Young braid group $Y_{(1,2)}$ (and also 4 orbits of B_3, which are 3 times bigger), but 5 orbits of the group P_3. Needless to say that there are 4 classes of the topological equivalence of polynomials.

We will not explain here how to construct the generators of the group Y_μ, but only remark that the replacing of P_k by Y_μ also leads to an acceleration of the algorithm.

(3) Finally, we would like to mention that we represent an orbit in a computer memory using a classical data structure of a *well balanced tree*, which permits us to construct an orbit of size N in $O(N \log N)$ time.

2.3 Results of Computations

For the degrees $n \leq 11$ there are 644 passports not covered by the general theorems of Sect. 1.4 and which therefore need a computer treatment. Among them there are 34 cases of non-unicity. They are summerized in Table 1. More details (concerning, for example, the orbit representatives or the monodromy groups) may be found in [9] (for the degrees $n \leq 9$) and in [3] (for the degree $n = 10$). Note that for the degree $n = 11$ the orbit is always unique for any passport, a result that is easy to state but which demanded long computational efforts.

Degree 12. The number of passports that need to be processed for the degree 12 is 833, which is bigger than for all the previous degrees taken together. The biggest putative orbit that needs to be computed corresponds to the passport $\pi = [4 \times 2^2 1^8, 3 \times 2^1 1^{10}]$ and is of size 102036672. This exceeds 50 times the memory of the biggest computer available to us. Furthermore, the estimated computation time of this orbit is about 2 years. It is clear that we need radically new ideas in order to treat *completely* the case $n = 12$. However, we do have partial results for the degrees 12 and bigger.

3 Invariants

3.1 Theorems of Ritt and Müller

We have already noted in Sect. 1.3 that the monodromy group is an invariant of the braid group action. The following classical result is due to Ritt [23] (1922).

Theorem 6. *The monodromy group of a polynomial covering is imprimitive if and only if the polynomial is a composition of non-linear polynomials of smaller degrees.*

A polynomial of a prime degree obviously cannot be a composition.

This theorem suggests considering separately primitive and imprimitive cases. The following remarkable result due to Müller[2] [20] is one of the consequences of the classification of finite groups.

Theorem 7. *Let P be a polynomial of degree n with $k \geq 3$ critical values and with* primitive *monodromy group* not equal to A_n or S_n. *Then $k = 3$, and there are only three cases possible:*

1. $n = 7$: $\pi = [3 \times 2^2 1^3] = [3 \times 2A]$, $G = PSL_3(2) \cong PSL_2(7)$;
2. $n = 13$: $\pi = [3 \times 2^4 1^5] = [3 \times 2A]$, $G = PSL_3(3)$;
3. $n = 15$: $\pi = [2^6 1^3, 2 \times 2^4 1^7] = [2B, 2 \times 2A]$, $G = PSL_4(2) \cong A_8$.

Here $2A$ and $2B$ is the notation of conjugacy classes used in the Atlas of Finite Groups [5]. In the next section we consider these three exceptional passports.

[2] We learned this result from Adrianov, who found it independently of Müller (private communication, February 1996)

Table 1. The list of polynomial passports up to degree 11 with a non unique orbit

	n	k	Passport	# of orbits	Orbit sizes	Reason of non-unicity
1	6	3	$[2 \times 2^2 1^2, 21^4]$	2	12, 3	Symmetry
2	7	3	$[3 \times 2^2 1^1]$	4	21, 21, 7, 7	Exceptional passport
3	8	3	$[3^2 1^2, 2^2 1^4, 21^6]$	2	28, 4	Symmetry
4	8	3	$[2^4, 2^2 1^4, 21^6]$	2	4, 2	Symmetry
5	8	3	$[41^4, 2 \times 2^2 1^4]$	2	48, 4	Symmetry
6	8	3	$[2^3 1^2, 2 \times 2^2 1^4]$	2	96, 8	Symmetry
7	8	4	$[3 \times 2^2 1^4, 21^6]$	2	992, 16	Symmetry
8	9	3	$[2 \times 2^3 1^3, 31^6]$	2	99, 3	Symmetry
9	9	3	$[421^3, 2^3 1^3, 21^7]$	2	117, 3	Composition
10	9	3	$[2^4 1, 2^3 1^3, 21^7]$	2	27, 3	Composition
11	9	3	$[2 \times 2^3 1^3, 2^2 1^5]$	2	297, 3	Composition
12	9	4	$[2 \times 2^3 1^3, 2 \times 21^7]$	2	891, 9	Composition
13	10	3	$[4^2 2^2, 2^2 1^6, 21^8]$	2	50, 5	Symmetry
14	10	3	$[3^2 2^2, 2^2 1^6, 21^8]$	2	50, 5	Symmetry
15	10	3	$[2 \times 3^2 1^4, 21^8]$	2	60, 5	Symmetry
16	10	3	$[3^2 1^4, 2^4 1^2, 21^8]$	2	60, 5	Symmetry
17	10	3	$[2 \times 2^4 1^2, 21^8]$	2	60, 5	Symmetry
18	10	3	$[3^2 1^4, 41^6, 2^2 1^6]$	2	85, 5	Symmetry
19	10	3	$[2^4 1^2, 41^6, 2^2 1^6]$	2	85, 5	Symmetry
20	10	3	$[61^4, 2 \times 2^2 1^6]$	2	120, 5	Symmetry
21	10	3	$[3^2 1^4, 2^3 1^4, 2^2 1^6]$	2	430, 15	Symmetry
22	10	3	$[2^4 1^2, 2^3 1^4, 2^2 1^6]$	2	430, 15	Symmetry
23	10	3	$[42^2 1^2, 2 \times 2^2 1^6]$	2	730, 10	Symmetry
24	10	3	$[3^2 21^2, 2 \times 2^2 1^6]$	2	730, 10	Symmetry
25	10	4	$[3^2 1^4, 2 \times 2^2 1^6, 21^8]$	2	3050, 25	Symmetry
26	10	4	$[2^4 1^2, 2 \times 2^2 1^6, 21^8]$	2	3050, 25	Symmetry
27	10	4	$[41^6, 3 \times 2^2 1^6]$	2	4275, 25	Symmetry
28	10	4	$[2^3 1^4, 3 \times 2^2 1^6]$	2	21400, 75	Symmetry
29	10	5	$[4 \times 2^2 1^6, 21^8]$	2	150000, 125	Symmetry
30	10	3	$[2^5, 31^7, 2^2 1^6]$	2	5, 2	Composition
31	10	3	$[2^5, 321^5, 21^8]$	2	10, 2	Composition
32	10	3	$[42^3, 2^2 1^6, 21^8]$	2	25, 10	Composition
33	10	4	$[2^5, 2^2 1^6, 2 \times 21^8]$	2	50, 20	Composition
34	10	3	$[2^5, 2 \times 2^2 1^6]$	4	10, 10, 5, 2	Composition and symmetry

3.2 Exceptional Passports

Degree 7. Passport: $\pi = [3 \times 2^2 1^3]$. The total number of the cacti is 56. There are four orbits:

- two orbits of size 21 with monodromy group A_7;
- and two orbits of size 7 with the monodromy group $PSL_3(2)$.

Degree 13. Passport: $\pi = [3 \times 2^4 1^5]$. The total number of cacti is 35672. There are five orbits:

- one orbit of size 35620 with the monodromy group A_{13};
- and four orbits of size 13 with the monodromy group $PSL_3(3)$.

Remark 8. It is very easy to find a representative of the A_{13}-orbit: practically any randomly chosen cactus will do. In order to compute the corresponding orbit of size 35620, one needs already a sufficiently powerful program: the first version of our program which did not use the well balanced trees was unable to do that. The most difficult part is however to find the representatives of the remaining four small orbits. This information may be found in [3].

Degree 15. Passport: $\pi = [2^6 1^3, 2 \times 2^4 1^7]$. The total number of cacti is 126000. There are four orbits:

- one orbit of size 125945 with the monodromy group A_{15};
- two orbits of size 5 with the monodromy group $PSL_4(2)$;
- and one "imprimitive" (or composition) orbit of size 45, with the monodromy group $(S_3 \wr S_5) \cap A_{15}$.

Groups Containing a Long Cycle. One may ask why there exist several orbits having the same monodromy group? The answer was found by G. Jones (1998). In fact, the monodromy group $G = <g_1, \ldots, g_k>$ is an invariant not only as an abstract group, but also as a particular permutation group, that is, as a particular subgroup of S_n. The same group G may have several conjugate copies inside S_n. But we must not forget either that we have fixed the product $g_1 \ldots g_k = (1, 2, \ldots, n) \in G$ once and for all.

Proposition 9. *There are two conjugate copies of the group $PSL_3(2)$ inside S_7 that contain the permutation $(1, 2, \ldots, 7)$ in their intersection; four copies of $PSL_3(3)$ inside S_{13} that contain $(1, 2, \ldots, 13)$; and two copies of $PSL_4(2)$ inside S_{15} that contain $(1, 2, \ldots, 15)$.*

The details of this construction will be explained elsewhere.

Remark 10. The reason why for the passport $[3 \times 2^2 1^3]$ there are two A_7-orbits remains mysterious. Up to now, this is the only case when we cannot propose a clear combinatorial invariant that would explain such a splitting. Just mention that the orbits themselves are not isomorphic: there is no bijection between them that would commute with the braid group action.

3.3 Splitting Types in A_n

As we have mentioned in Sect. 1.3, the set of the conjugacy classes of the permutations g_i is invariant under the braid group action. The conjugacy classes in S_n are completely determined by the cyclic structure of their elements. For the group A_n this is not the case. The following lemma may be found in [15] (Lemma 1.2.10):

Lemma 11. *A set of permutations of a given cyclic structure splits into two conjugacy classes in A_n if and only if the lengths of all cycles are odd and different.*

Note that the parity of the monodromy group is easily seen from the passport: the monodromy group G is a subgroup of A_n if and only if the sum $\sum_{j=1}^{p_i}(d_{ij}-1)$ is even for all $i = 1, 2, \ldots, k$.

The "smallest" example that may be constructed using the idea of the above lemma corresponds to the passport $\pi = [97531, 2 \times 31^{22}]$ ($n = 25$). Indeed, there are two orbits for this passport, each of size 300.

3.4 Imprimitive Case

Let us call a passport *decomposable* if there exist polynomials having this passport which are compositions of non-linear polynomials of smaller degrees. The property of the decomposability of a passport may be verified using only the passport itself, without constructing the corresponding cacti. It is less obvious to verify if, along with decomposable cacti, there also exist the indecomposable ones having the same passport; all the procedures we know of are rather complicated.

The important combinatorial data is supplied by the *composition type*. We don't have enough space to discuss this notion here. Therefore we only mention that for a polynomial $P = Q \circ R$ the composition type describes how the critical values of R are related to the critical points of Q.

In the imprimitive case the knowledge of the monodromy group is not at all sufficient for distinguishing between the different orbits. Let us take, for example, the passport $\pi = [2^6, 321^7, 2^2 1^8]$ ($n = 12$). There are 6 orbits for this passport, and for all of them the monodromy group is $G = S_6 \wr S_2$. Note that, contrary to Prop. 9, this group is also unique as a particular subgroup of S_{12}. Indeed, the presence of the long cycle $(1, 2, \ldots, 12)$ implies the unique choice of the two imprimitivity blocks of size 6, namely, $\{1, 3, 5, 7, 9, 11\}$ and $\{2, 4, 6, 8, 10, 12\}$. The true reason of the splitting is the fact that there are 4 different composition types, and also for one of these types there are 3 classes of polynomials R.

Another possibility is illustrated by the passport $\pi = [2^5, 4 \times 21^8]$. In spite of the fact that this passport is decomposable, there is a single orbit here, because all the corresponding compositions are of the same type (and there are no indecomposable cacti for this passport).

Finally we must mention that even if there is a single class of the topological equivalence of polynomials Q, and the same is valid for R, and if we take

their compositions of the same type, there may nevertheless be several classes of polynomials $P = Q \circ R$. One such example is discussed in detail in [3].

Despite all these complications we may say that the study of the imprimitive case is to a large extent reduced to the classification of the factors, that is, mainly to the primitive case.

Remark 12. Symmetry is a particular case of composition, namely, when the polynomial $R(x) = x^m$. However, in Table 1 we distinguish these two situations and use the word "composition" only in the case of a composition more complicated than a simple symmetry.

4 Conjectures and Perspectives

4.1 Conjectures S_n and A_n

The following conjectures were formulated during our discussions with A. Khovanskii. Let us say that a polynomial passport is *of type* S_n if it is indecomposable and odd (that is, if among g_i there exists at least one odd permutation). Theorems of Sect. 3.1 imply that for such a passport the only possible monodromy group is S_n. It is also easy to see that every polynomial passport with $k > (n+1)/2$ is of type S_n.

Conjecture 13. *For any passport of type S_n there exists a single orbit of the braid group action.*

Let us say that a polynomial passport is *of type* A_n if it is indecomposable, even (that is, all the permutations g_i are even) and different from the three exceptional passports of Sect. 3.2. Theorems of Sect. 3.1 imply that for such a passport the only possible monodromy group is A_n.

Conjecture 14. *For a passport of type A_n there are two possibilities:*

1. *If there is no splitting type partitions in the passport (like in Sect. 3.3), then there exists a single orbit.*
2. *If there exists a splitting type in the passport, then there exist exactly two orbits.*

(Simple arithmetic considerations show that there could exist at most one splitting type in an even polynomial passport.)

If these two conjectures turn out to be true, the problem will be entirely reduced to the study of the imprimitive case.

4.2 Meromorphic Functions with a Single Pole

After the polynomial case, what should be the next case to study? Previously our project (outlined in [9]) was to attack the topological classification of the rational functions. Today we think that a more "convenient" class of functions is

the class of meromorphic functions defined on Riemann surfaces of genus $g > 0$ and having a single pole. Combinatorially this would mean that we preserve condition (2) but not condition (3).

The reasons for this change of the intention are the following:

(1) Many particular features of our programs are based on the fact that the product $g_1 \dots g_k = c = (1, \dots, n) \in S_n$ is a cyclic permutation. The case of an arbitrary permutation would necessitate a re-programming of almost everything, while the case of a positive genus needs only very little adjustment.

(2) There are some enumerative results for the factorizations of a long cycle in a product of permutations of given cyclic structures [12], while the similar results for an arbitrary permutation are very scarce.

(3) An approach based on the formula of Frobenius, that uses the characters of the symmetric group, permits in principle to compute the number of above factorizations for an arbitrary permutation. But this method has another flaw: it counts also "non-transitive cacti" (that is, the sequences $[g_1, \dots, g_k]$ that generate non-transitive subgroups of S_n), and therefore necessitates a cumbersome inclusion-exclusion procedure, while the presence of a long cycle automatically guarantees the transitivity.

(4) Last but not least, the permutation groups containing a long cycle are classified [10] while the "planar monodromy groups" are not [13].

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