

Plane and projective meanders

S.K. Lando

Institute of New Technologies, Kirovogradskaya 11, 113587 Moscow, Russia

A.K. Zvonkin

*Cybernetics Council, Russian Academy of Sciences, Vavilova 40, 117333 Moscow, Russia, and
LaBRI, Université Bordeaux I, 351 cours de la Libération, 33405 Talence Cedex, France*

Abstract

Lando, S.K. and A.K. Zvonkin, Plane and projective meanders, Theoretical Computer Science 117 (1993) 227–241.

This paper considers combinatorial properties of meanders – nonself-intersecting curves on the euclidean and projective plane. Both theoretical and numerical results are given, and various connections with other domains in mathematics are indicated.

Résumé

Lando, S.K. and A.K. Zvonkin, Plane and projective meanders, Theoretical Computer Science 117 (1993) 227–241.

Cet article considère les propriétés combinatoires des méandres – des courbes sans self-intersections sur le plan euclidien ou projectif. Des résultats théoriques et numériques sont présentés, des liens divers avec les domaines différents des mathématiques sont indiqués.

0. Introduction

J'errais dans un méandre;
J'avais trop de partis, trop compliqués, à prendre ...
— E. Rostand, *Cyrano de Bergerac*, acte 1, scène 5.

A highway from west to east crosses several times a river flowing from south–west to east. Enumerate the bridges as they are located along the highway (from west to east). The order of the bridges along the river determines a permutation. Following Arnol'd, we call the permutation (and a corresponding geometrical image) *a meander*.

Correspondence to: A.K. Zvonkin, LaBRI, Université de Bordeaux I, 351 cours de la Libération, 33405 Talence Cedex, France.

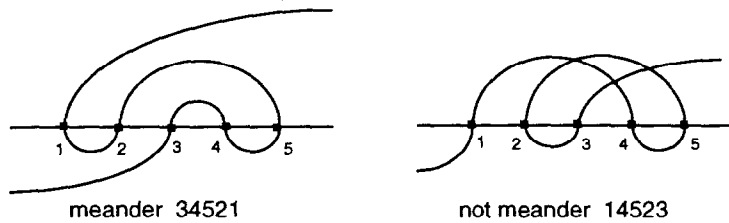


Fig. 1.

Obviously, not all permutations can be obtained in this way. In particular, in meanders even numbers must occupy even positions, odd numbers, odd positions (see Fig. 1).

Numerous pictures of meanders can be found in the last paper by Henri Poincaré “Sur un théorème de géométrie” [9], where he tried to prove, by means of meanders, that a transformation of a ring into itself preserving the area and shifting border circles into opposite directions has not less than two fixed points. The theorem was proved by Birkhoff in 1913 by a different method, but its generalization on the transformation of a sphere with handles was proved by Eliashberg [3] in 1978 with the help of meanders. “Projective meanders” to be defined below were used by Arnol’d [1] as a tool for analyzing differential-geometric properties of the manifold of zeroes of hyperbolic polynomials. In a number of papers meanders appeared not so much as a tool but as an object of investigation. For instance, in [10] “plane permutations” are introduced and investigated that coincide with “closed meanders” to be defined below. Such permutations prove to be sorted in linear time. In [7] a class of mazes is introduced that are in one-to-one correspondence with meanders.

The problem of enumerating meanders proved to be especially complicated. In [5] for a similar problem of enumeration for the number of folding a strip of stamps certain recurrence formulas are introduced. They can serve as the basis for constructing an algorithm of computation of corresponding numbers but, unfortunately, they do not yield either explicit formulas or even the information on the asymptotics of the number sequence in question, while the algorithm has exponential complexity and does not allow us to compute a large number of sequence terms. In a paper [6] of present authors the problem of enumerating closed meanders was studied. We established nontrivial upper estimates for the main term of the asymptotics, and we observed a relationship between the problem of meanders and the theory of formal languages and that of Feynman diagrams in the quantum field theory. Cori [2] attracted the authors’ attention to the fact that the problem of enumerating closed meanders was equivalent to that of determining the complexity of a class of hypermaps.

Thus, the problem of meanders seems to belong to the simply formulated but fairly difficult problems of combinatorial analysis related to different sections of mathematics and is a touchstone for various methods of enumerative combinatorics.

Table 1
Table of meandric numbers

n	0	1	2	3	4	5	6	7	8	9	10
m_n	1	1	1	2	3	8	14	42	81	262	538
n	11	12	13	14	15	16					
m_n	1828	3926	13 820	30 694	110 954	252 939					
n	17	18	19	20	21						
m_n	933 458	2 172 830	8 152 860	19 304 190	73 424 650						
n	22	23	24	25	26	27					
m_n	?	678 390 116	?	6 405 031 050	?	61 606 881 612					

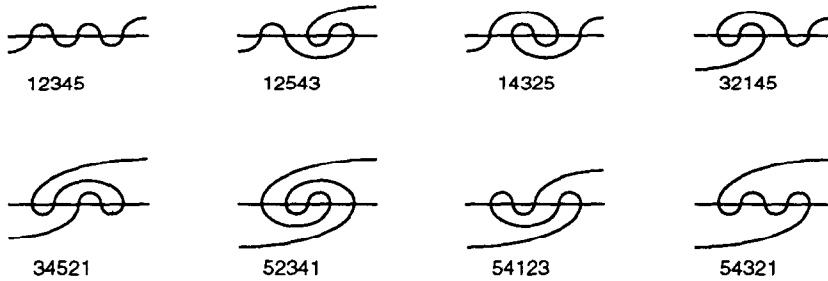


Fig. 2.

The present paper examines arithmetical properties of meandric numbers and also introduces and studies projective meanders.

1. Arithmetical properties of meandric numbers

Definition 1.1. Denote the number of meanders passing through n bridges by m_n , $n = 1, 2, 3, \dots$. We call m_n the n th meandric number. Put $m_0 = 1$. Sequence m_n can be readily shown to increase monotonically when $n \geq 2$ (see Table 1).

Table 1 is based on computational results obtained by the present authors, Phillips [7], Reeds and Shepp (ibid)¹. With a slight modification the algorithm discussed in [6], which enumerates closed meanders, also allows one to enumerate open meanders.

Figure 2 shows all the eight meanders of order 5.

¹ S. Plouffe [8] communicated to us two more meandric numbers:

$$m_{29} = 602\,188\,541\,928. \quad m_{31} = 5\,969\,806\,669\,034.$$

Calculations belong to V.R. Pratt.

Statement 1.2. *The number m_n is odd iff $n=2^q$, $q=0, 1, 2, 3, \dots$*

To prove this we shall need the following lemma.

Lemma 1.3. $m_{2n} \equiv m_n \pmod{2}$, $n=1, 2, \dots$

Proof. On a set of meanders of order n define the involution of “reflection” when permutation (a_1, \dots, a_n) corresponds to permutation (a'_1, \dots, a'_n) , $a'_i = n+1 - a_i$, $i=1, \dots, n$. Geometrically, this is the reflection with respect to the vertical axis passing through the middle of segment $[1, n]$ (when n is odd it remains to “correct” the directions of the curve ends) (see Figs. 3, 4).

To each symmetric meander of order $2n$ one can put into correspondence a meander of order n , namely, its left-hand half. As to nonsymmetric meanders, they are divided into pairs, which proves the lemma. \square

Proof of Statement 1.2 (conclusion). This now follows from the fact that $m_1 = 1$; for all the remaining odd orders k the number m_k is even, since the involution of reflection on a set of meanders of an odd order does not have fixed points. \square

1.1. Closed meanders

By joining the ends of the meandric curve passing through an even number of bridges, we obtain a *closed meander* (see Fig. 5).

Different meanders passing through $2n$ bridges may correspond to the same closed meander (see Fig. 6).

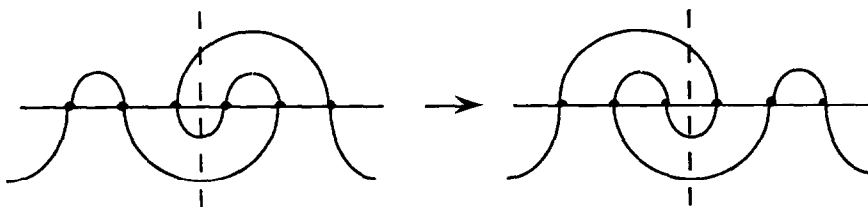


Fig. 3.

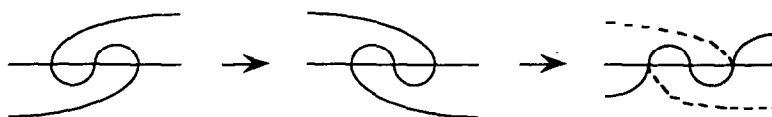


Fig. 4.

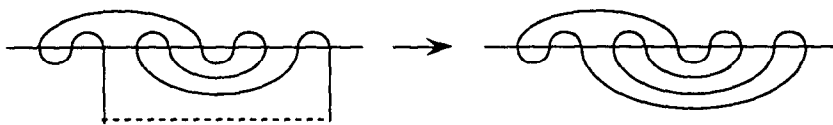


Fig. 5.



Fig. 6.

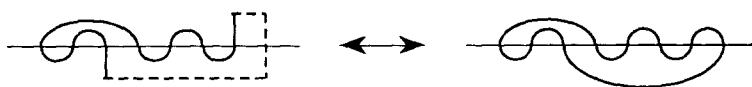


Fig. 7.

The number of closed meanders passing through $2n$ bridges will be denoted by M_n .

Statement 1.4. $M_n = m_{2n-1}$.

Indeed, there is a natural one-to-one correspondence between meanders passing through $2n - 1$ bridges and closed meanders passing through $2n$ bridges. This can be seen in Fig. 7.

Thus, numbers $M_n = 1, 2, 8, 42, \dots$ are actually contained in Table 1.

Statement 1.5 (Lando and Zvonkin [6]). *If $n = p^q$, where p is a prime, $q > 1$, then $M_n = m_{2n-1} \equiv 2 \pmod{p}$.*

1.2. Systems of closed meanders and their distribution according to the number of components

If we omit the condition of connectedness of the curve in the definition of a closed meander, we will obtain the definition of a *system of meanders*. Note that systems of meanders are in one-to-one correspondence with the pairs of correct parenthesis systems: the set of arcs in the upper half plane corresponds to one parenthesis system and the set of arcs in the lower half plane to the other parenthesis system. Consequently, the number of meander systems of order n is equal to the square of the n th Catalan number. In Fig. 8 the meander system of order 7 with 3 components is shown.

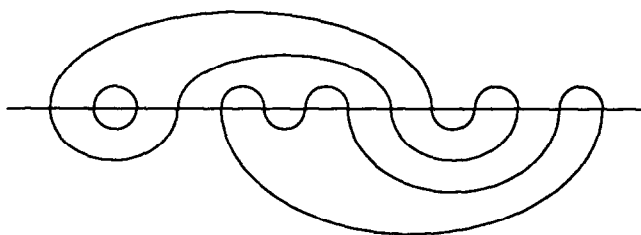


Fig. 8.

Table 2

$k \backslash n$	1	2	3	4	5	6	7
1	1	2	8	42	262	1828	13 820
2		2	12	84	640	5236	45 164
3			5	56	580	5894	60 312
4				14	240	3344	42 840
5					42	990	17 472
6						132	4004
7							429

In Table 2 the distribution of meander systems according to the number of components is given (calculations of D. Ivanov). The order of a system is denoted by n , and k denotes the number of components.²

1.3. Distribution of meandric numbers according to the position of the first bridge

Denote by $m_{n,k}$ the number of meanders with n bridges for which the position of the first bridge is k . It is obvious from Fig. 2 that $m_{5,1} = 3$, $m_{5,3} = 2$, $m_{5,5} = 3$. In Table 3 we give values for number $m_{n,k}$ for $n = 1, 2, \dots, 18$.

Statement 1.6. (1) $m_{n+1,1} = m_n$; $m_{2k-1,1} = m_{2k,3}$.

(2) When n is odd, sequence $m_{n,1}, m_{n,3}, \dots, m_{n,n}$ is symmetric, i.e. $m_{n,k} = m_{n,n+1-k}$ when $k = 1, \dots, n-1$.

(3) When n is even, sequence $m_{n,3}, m_{n,5}, \dots, m_{n,n-1}$ is symmetric, i.e. $m_{n,k} = m_{n,n+2-k}$ when $k = 3, \dots, n-1$.

²Using a software based on a system of symbolic calculations, Plouffe [8] guessed the generating function for the sequence 2, 12, 56, 240, 990, 4004, ... of near-diagonal terms:

$$f(t) = \frac{32}{\sqrt{1-4t}(1+\sqrt{1-4t})^4}.$$

This formula was later proved by Zeilberger [8].

Table 3

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	2	3	8	14	42	81	262	538
3			1	1	2	3	7	14	36	81	221
5					3	3	7	11	28	57	155
7							14	14	36	57	155
9									81	81	221
11											538
$k \backslash n$	12	13	14	15	16	17	18				
1	1828	3926	13820	30694	110954	252939	933458				
3	538	1530	3926	11510	30694	92114	252939				
5	353	1003	2458	7214	18575	55880	149183				
7	316	902	2053	6059	14810	44842	115009				
9	353	1003	2053	6059	13827	41908	102555				
11	538	1530	2458	7214	14810	44842	102555				
13		3926	3926	11510	18575	55880	115009				
15				30694	30694	92114	149183				
17						252939	252939				

All the above statements are proved by establishing a one-to-one correspondence between the meandric families under consideration. Thus, for instance, in proving statements (2) and (3) the reflection operation defined is made use of.

Conjecture 1.7. For any n sequence $m_{n,1}, m_{n,3}, \dots$ is unimodal i.e. there can be found a k such that

$$m_{n,1} \geq m_{n,3} \geq \dots \geq m_{n,k} \leq m_{n,k+2} \leq \dots$$

2. Projective meanders

2.1. Definition

Consider $2n$ points on a circle that divide it into equal arcs. Enumerate them in succession by numbers $1, 2, \dots, 2n$. Now divide the points into n pairs so that the chords connecting points in each pair would not intersect. Identify the diametrically opposite points of the circle, thus turning the disc into a projective plane. Then the set of chords forms a family of closed non-intersecting curves on a projective plane. We shall call the set of curves a *system of projective meanders* of order n . If the family consists of a single curve, we shall call the latter a *projective meander*. The number of projective meanders of order n will be denoted by pm_n .

As is known, the number of ways of drawing n non-intersecting chords and, consequently, the number of systems of projective meanders of order n is equal to the n th Catalan number $Cat_n = (1/(n+1)) \binom{2n}{n}$.

In Fig. 9 one can see all the five systems of meanders of order 3; the two upper ones are projective meanders.

The number of projective meanders in the system is actually equal to the number of cycles in a permutation on the set of $2n$ elements, the permutation being defined by means of two involutions without fixed points: one involution is given by a system of chords, the other one, by central symmetry $k \rightarrow k + n \pmod{2n}$.

In Table 4 the values of projective meandric numbers pm_n for $n=0, 1, \dots, 15$ are given (computed by the present authors).

The algorithms of polynomial complexity for computing meandric and projective meandric numbers are not yet known.

Statement 2.1. Sequences pm_0, pm_2, pm_4, \dots and pm_1, pm_3, pm_5, \dots increase monotonically.

The proof is based on the fact that it is possible to make from any projective meander of order n one or more projective meanders of order $n+2$ by means of an

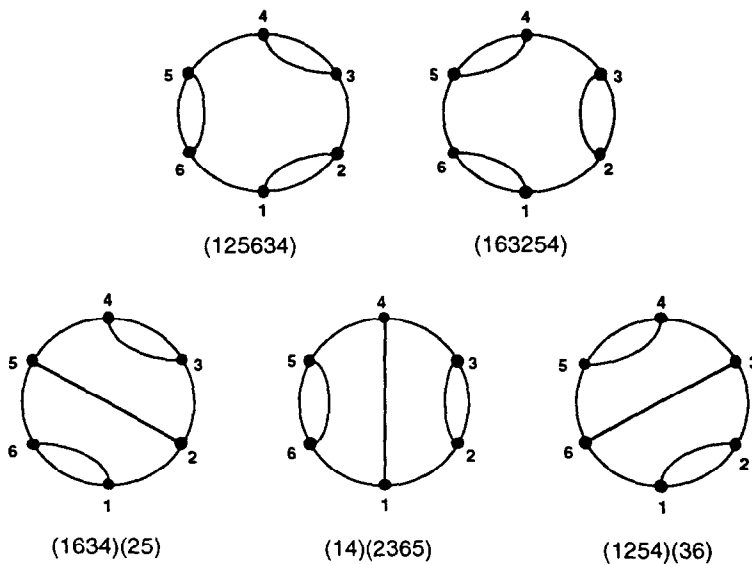


Fig. 9.

Table 4

n	0	1	2	3	4	5	6	7	8	9	10
pm_n	1	1	2	2	8	12	52	86	400	710	3404
n	11	12		13		14		15			
pm_n	6316	30 888		59 204		293 192		576 018			

operation of “stretching”, to be defined below. The projective meanders of order $n + 2$ thus obtained will be different.

For convenience, we shall further draw an infinitely far straight line on a projective plane as a horizontal one. The operation of stretching consists in the following: (1) cut one of the “upper” arcs; (2) add points $2n + 1, 2n + 2$ and join them to the ends of the cut arc; (3) add two points between points n and $n + 1$ and join them by an arc; re-enumerate the points (see Fig. 10).

2.2. Action of group \mathbb{Z}_{2n} and its orbits

Group \mathbb{Z}_{2n} acts on a set of systems of closed meanders of order n and on a set of systems of projective meanders. Its generator is given by the cyclic shift $k \rightarrow k + 1 \pmod{2n}$. The action preserves the number of components. Investigation of the orbits of the action allows us to receive some congruences for meandric numbers, see, for example, Statement 1.5.

Statement 2.2. (1) If $n = p^q$ is a power of an odd prime then $pm_n \equiv 2 \pmod{2p}$.
 (2) If $n = 2^q$, then $pm_n \equiv 0 \pmod{2n}$.

In order to prove the above statement we shall need the following lemma.

Lemma 2.3. Let $n > 2$. Then the order of the orbit of group \mathbb{Z}_{2n} action on the set of projective meanders of order n does not divide n .

Proof. If the orbit order divides n , the system of arcs over the straight line passes into itself under the shift $k \rightarrow k + n \pmod{2n}$. If there is at least one arc whose beginning and end are among the first n points, it is isolated into a separate meander together with the arc shifted for n (Fig. 11). If there is no such arc, the farthest outside and inside arcs are isolated into a separate meander (Fig. 11).

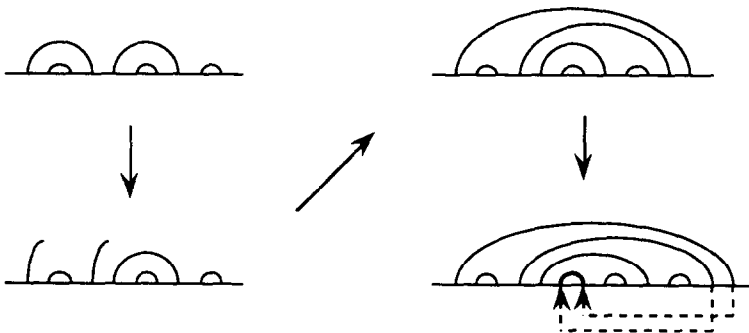


Fig. 10.

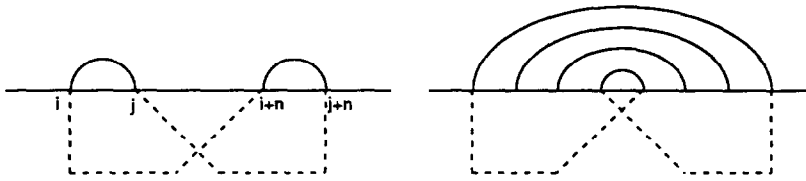


Fig. 11.



Fig. 12.

Proof of Statement 2.2 (conclusion). (1) Orbit orders divide $2n = 2p^q$. Therefore, they are either equal to 2 or are divisible by $2p$. There exists only one orbit of order 2 forming a projective meander (Fig. 12).

(2) The order of any orbit divides $2n = 2^{q+1}$ and does not divide 2^q . Therefore, it equals 2^{q+1} . \square

It is well-known that systems of nonintersecting chords are in one-to-one correspondence with rooted plane trees, and their orbits under the action of \mathbb{Z}_{2n} correspond to (nonrooted) plane trees. Some plane trees correspond to projective meanders, others correspond to the systems of projective meanders. But we do not dispose of a geometric criterion to distinguish between these two cases.

3. Estimates of asymptotics

3.1. Closed meanders

We call a system of closed meanders *irreducible* if there is no subsegment $[a, a + 1, \dots, b] \subset [1, 2, \dots, 2n]$ such that through its points there passes an independent system of meanders. Denote the number of irreducible systems of closed meanders passing through $2n$ points by N_n .

Any single meander obviously forms an irreducible system so that $M_n \leq N_n$. The left-hand figure (see Fig. 13) shows an irreducible system, the right-hand figure, a reducible one.

In [6] the present authors obtained the following results.

Theorem 3.1. (1) *The generating function $N(x) = \sum_{n=0}^{\infty} N_n x^n$ for the number of irreducible systems of meanders satisfies the functional equation*

$$B(x) = N(xB^2(x)), \tag{1}$$

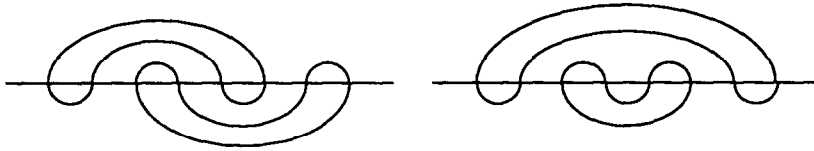


Fig. 13.

Table 5

n	0	1	2	3	4	5	6	7	8
N_n	1	1	2	8	46	322	2546	21 870	199 494

where $B(x)$ is a generating function for square Catalan numbers:

$$B(x) = \sum_{n=0}^{\infty} (\text{Cat}_n)^2 x^n.$$

(2) The function $B(x)$ is expressed by the formula

$$B(t^2) = \frac{1}{4t^2} \left(-1 + \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - 8t \cos \phi + 16t^2} d\phi \right).$$

(3) The radius of convergence of the series $N(x)$ equals $[(4 - \pi)/\pi]^2 = 1/13.3923 \dots$

Empirical estimation of the ratio M_{n+1}/M_n obtained by means of the Padé approximation yields the value 12.26 ...

Equation (1) allows us to construct a polynomial algorithm for computing the numbers N_n . We give in Table 5 a few initial values:

If $\mathcal{B}(x, u)$ is a generating function for the system of meanders classifying them by the number of irreducible components, then by the methods similar to that in [6], it is easy to obtain the equation

$$\mathcal{B}(x, u) = N(xu(\mathcal{B}^2(x, u))). \tag{2}$$

3.2. Projective meanders

We shall encode a system of projective meanders of order n or, which is the same, a system of n nonintersecting arcs in the upper semi-plane by a word of n letters in the alphabet $\{a, b, c, d\}$ according to the following rule. Each point is either the beginning or the end of an arc. Consider points i and $i+n$; the

ith letter of the word will correspond to them. This letter is defined by the following rule:

(beginning, beginning)	$\rightarrow a$
(beginning, end)	$\rightarrow b$
(end, beginning)	$\rightarrow c$
(end, end)	$\rightarrow d$

The example is given in Fig. 14.

It can be seen that the number of all words of length n in the alphabet $\{a, b, c, d\}$ equals 4^n while the number of systems of projective meanders of order n , i.e. the n th Catalan number, is asymptotically equal to $(1/\sqrt{\pi}) 4^n n^{-3/2}$.

It is possible to put into correspondence with each system of closed meanders of order n a system of projective meanders of order $2n$ with the same number of components. To achieve this, the following operations should be carried out with the system of closed meanders: (1) “cut off” the lower system of arcs and transfer it by a turn of 180° into the upper semiplane; (2) apply the reflection operation to the right-hand half of the system of arcs thus obtained (see Fig. 15). Therefore, we receive out of closed meanders of order n projective meanders of order $2n$, though not all of them but only those whose system of arcs is divided into two halves: one “inhabits” the set of points $[1, 2, \dots, n]$, the other, $[n, n+1, \dots, 2n]$. We have proved the following statement.

Statement 3.2. $pm_{2n} \leq M_n$.

Note that encoding of systems of closed meanders by words in the alphabet $\{a, b, c, d\}$ adopted in [6] and encoding of systems of projective meanders adopted in the present paper are consistent.

We shall further need the following lemma (see, for example, [4], i. 2.8.8).

Lemma 3.3. *Let the set A of words in the alphabet of k letters possess the property that any two words $u, v \in A$, $u \neq v$ “do not overlap”, i.e. none of them is a subword of another one and there do not exist three words α, β, γ with a nonempty β such that $u = \alpha\beta$, $v = \beta\gamma$. Let $f_A = \sum a_n x^n$ be a generating function for the words of the set A , i.e. a_n is the number of*

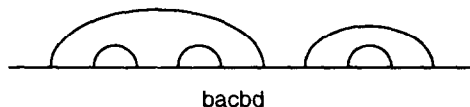


Fig. 14.

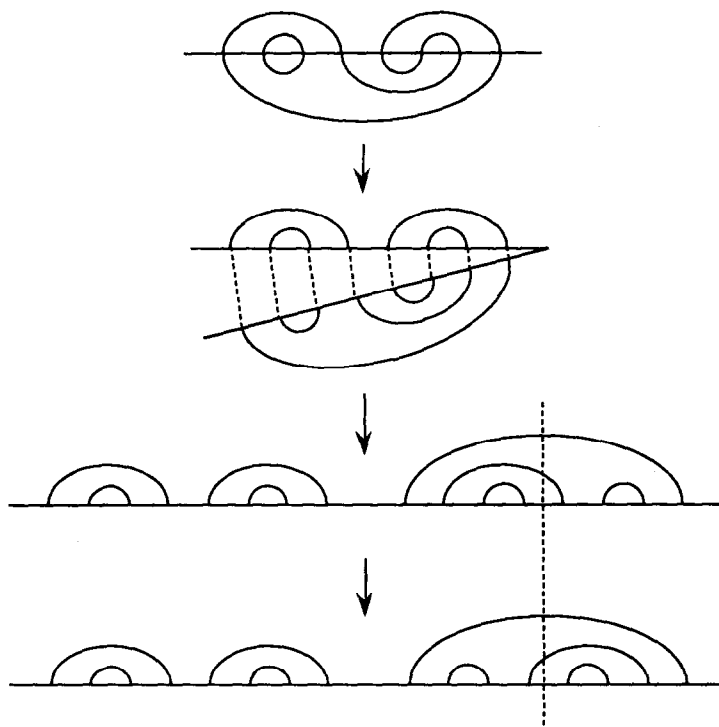


Fig. 15.

words of length n belonging to the set A . Then the generating function $F(x)$ for the set of all words in the same alphabet not containing a single subword from A equals

$$F_A(x) = (1 - kx + f_A)^{-1}.$$

For instance, the generating function for the words in the alphabet $\{a, b, c, d\}$ not containing a single subword ad equals $(1 - 4x + x^2)^{-1}$.

Theorem 3.4. For all $\varepsilon > 0$ and all large n

$$pm_{2n} \leq \left(\frac{1}{R_0} + \varepsilon \right)^{2n},$$

where R_0 is the least positive root of function $1 - 4x + (N(x^2) - 1)$, and $N(x)$ is the generating function for irreducible systems of closed meanders.

Proof. We have to estimate from below the convergence radius of the series $\sum_{n=0}^{\infty} pm_n x^n$. The radius will obviously remain unchanged if we change in an arbitrary way a few initial coefficients of the series.

For obvious geometrical reasons the set of words in the alphabet $\{a, b, c, d\}$ describing irreducible systems of closed meanders satisfy the nonoverlapping condition formulated in the lemma. Let A_m be a set of such words of length not more than $2m$, except the empty word. Then a set of all words in the same alphabet, not containing a single subword from A_m and with a length of more than $2m$, contains all the words describing projective meanders (as well as many unnecessary words). Thus, coefficients of the corresponding generating function

$$F_m(x) = (1 - 4x + f_{A_m})^{-1}$$

grow not slower than the numbers pm_n , the convergence radius of the series $F_m(x)$ being equal to the least positive root R_m of the polynomial

$$1 - 4x + f_{A_m}(x).$$

Increasing m , we increase the number of forbidden subwords and, thus, improve our estimate, leaving asymptotically fewer words of a large length (see Fig. 16).

Polynomials $1 - 4x + f_{A_m}(x)$ converge coefficient-wise to the series $1 - 4x + (N(x^2) - 1)$ (x^2 instead of x appears because meanders of order n are described by words of length $2n$; unity is subtracted from $N(x^2)$ due to the fact that the empty word does not enter the number of forbidden ones). Since the coefficients of the series $N(x)$ are positive, $R_m \nearrow R_0$, where R_0 is the root of function

$$1 - 4x + (N(x^2) - 1).$$

The theorem is proved. \square

A specific feature of the method is that the more the number of terms of the series $N(x)$ that we compute, the more accurate is the estimate of the rate of growth for numbers pm_n .

Our calculations give the value $(1/R_0)^2 = 13.42 \dots$

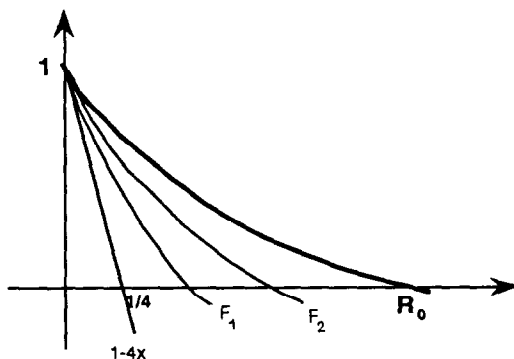


Fig. 16.

Acknowledgments

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